Weak approximation of averaged diffusion processes

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Abstract

We derive expansion results in order to approximate the law of the average of the marginal of diffusion processes. The average is computed w.r.t. a general parameter that is involved in the diffusion dynamics. Our approximation is based on the use of proxys with normal distribution or log-normal distribution, so that the expansion terms are explicit. We provide non asymptotic error bounds, which justifies the expansion accuracy as the time or the diffusion coefficients are small in a suitable sense.

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Introduction. Let T be a positive real number and consider a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, which supports a q-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$. Here, $(\mathcal{F}_t)_{0\le t\le T}$ is the usual $\mathbb{P}$-augmented natural filtration of W.

To a non empty set of parameters $\mathcal{A}$, we associate a family of scalar diffusion processes $\{X^\alpha : \alpha \in \mathcal{A}\}$: for each $\alpha \in \mathcal{A}$, X^α solves the stochastic differentiable equation (SDE in short)

$$X_t^\alpha = X_0^\alpha + \int_0^t b^\alpha(s, X_s^\alpha)\mathrm{d}s + \int_0^t \sigma^\alpha(s, X_s^\alpha)\mathrm{d}W_s. \tag{0.1}$$

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The initial value X_0^α is finite and non-random. The coefficients b^α and σ^α are supposed to be Lipschitz continuous w.r.t. x, for each $\alpha \in \mathcal{A}$. Thus, for each $\alpha \in \mathcal{A}$ there is a unique strong solution on $[0, T]$ to (0.1), well defined except on a negligible set $\mathcal{N}^\alpha$. We assume that $\mathcal{A}$ is a countable set, so that the family of diffusion processes $\{X^\alpha : \alpha \in \mathcal{A}\}$ is simultaneously well defined with probability one (namely, on the set $(\cup_{\alpha\in\mathcal{A}}\mathcal{N}^\alpha)^c$). For the case of an uncountable set $\mathcal{A}$, see the discussions in Remark 1.1. Then, we consider a *finite positive* measure ν on $\mathcal{A}$: to have short notation, we often write an integral w.r.t. ν as

$$\nu(f^\cdot) := \int_{\mathcal{A}} f^\alpha \nu(da)$$

for any function f defined on $\mathcal{A}$. The purpose of this work is to provide weak approximation and expansion results regarding the *arithmetic mean*

$$\nu(X_T^\cdot) := \int_{\mathcal{A}} X_T^\alpha \; \nu(\mathrm{d}\alpha) \tag{0.2}$$

and the *mean of exponentials*

$$\nu^{\exp}(X_T^\cdot) := \int_{\mathcal{A}} \exp(X_T^\alpha) \; \nu(\mathrm{d}\alpha), \tag{0.3}$$

as the parameters b^α and σ^α or their derivatives are small, or the time T is small. Although we may write $\nu^{\exp}(X_T^\cdot) = \nu(\exp(X_T^\cdot))$ and that $(\exp(X_\cdot^\alpha))_{\alpha\in\mathcal{A}}$ is another family of SDEs, we reserve to the case (0.3) a specific analysis because $\nu^{\exp}(X_T^\cdot)$ takes positive values. The quantities to approximate have the form

$$\mathbb{E}[\varphi(\nu(X_T^\cdot))] \quad \text{and} \quad \mathbb{E}[\Phi(\nu^{\exp}(X_T^\cdot))], \tag{0.4}$$

for different test functions φ and Φ (satisfying various smoothness assumptions). For the arithmetic mean, normal approximations are provided, while for the mean of exponentials, we derive log-normal approximations (thus maintaining the positivity). Non asymptotic results are proved, emphasizing the role of coefficients in the approximation accuracy. Since the studied quantities are $\nu(X_T^\cdot)$ and $\nu^{\exp}(X_T^\cdot)$, up to a renormalization of ν we can assume that $\nu(\mathbf{1}_{\mathcal{A}}) = 1$; thus ν *is a probability measure.* Without further reference we always assume that $\int_{\mathcal{A}} \exp(|X_0^\alpha|)\nu(\mathrm{d}\alpha) < +\infty$. It is a sufficient condition to ensure enough integrability on $\nu(X_T^\cdot)$ and $\nu^{\exp}(X_T^\cdot)$, see Lemma 1.2.

Here are examples where averaged diffusion may be interesting for applications.

Example 0.1 (Time-average of a scalar diffusion process) *The discrete time-average on* $[0, T]$ *is defined by a set of* n *times:*

$$\mathcal{A} = \{0 < \alpha_1 < \cdots < \alpha_n \leq T\} \subset [0, T].$$

The probability measure ν *may be the uniform measure on* $\mathcal{A}$ *(coming for instance from the rectangle rule in numerical integration) or any other discrete measure (from* *the trapezoidal rule ...). The SDE of interest is given by*

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s.$$

To embed the time-average of this single scalar diffusion process into our framework, we set $X_t^\alpha = X_{\alpha\wedge t}$ *which coefficients are given*

$$b^\alpha(s,x) = \mathbf{1}_{s\le\alpha} b(s,x), \qquad \sigma^\alpha(s,x) = \mathbf{1}_{s\le\alpha}\sigma(s,x).$$

Indeed, we observe that $\int_{\mathcal{A}} X_T^\alpha \nu(\mathrm{d}\alpha) = \sum_{i=1}^n X_{\alpha_i\wedge T}\nu(\mathbf{1}_{\alpha_i}) = \sum_{i=1}^n X_{\alpha_i}\nu(\mathbf{1}_{\alpha_i})$.

The continuous time-average is similarly defined, by taking $\mathcal{A} = [0,T]$ *and* $\nu(\mathrm{d}\alpha) = \frac{1}{T}\mathbf{1}_{[0,T]}(\alpha)(\mathrm{d}\alpha)$ *for instance. In that case,* $\mathcal{A}$ *is an uncountable set, but notice that, nevertheless, the whole family* $(X^\alpha)_{\alpha\in\mathcal{A}}$ *is well defined with probability 1: hence, the expansion results below apply to this case.*

Some applications. *In Finance, Asian options are defined as a financial contract written on the average over a period of time of the price of a financial asset (modeled by* $(X_t)_{0\le t\le T}$ *or* $(\exp(X_t))_{0\le t\le T}$*). In 1987, Banker's Trust Tokyo office (which the name "Asian" option is originated of) used them for the first time for pricing average options on crude oil contracts; see [TW91] for details. In Random Mechanics [KS86],* X *may define the velocity of a system, on which random forces are applied. The integral* $\int_0^T X_t \mathrm{d}t$ *is then related to the (random) position of the system at time* T*. Another field of application may be glaciology, where the modeling of ice-core data can be made through an integrated diffusion process [DDA02].*

Example 0.2 (Mixture of diffusion model) *Assume that* $\mathcal{A}$ *is the set of the stocks entering in the definition of a Stock Index (CAC40 for instance), that the weights* $\nu(\mathbf{1}_{\alpha_i})$ *are equal to the capitalization-weights of the related companies and that* $(X^\alpha)_{\alpha\in\mathcal{A}}$ *represents the price or the log-price of the stocks of the index. Then* $\int_{\mathcal{A}} X_T^\alpha\nu(\mathrm{d}\alpha)$ *or* $\int_{\mathcal{A}}\exp(X_T^\alpha)\nu(\mathrm{d}\alpha)$ *is the value of the index at time* T*. Thus approximating the law of the averages is relevant in financial engineering, for pricing contracts where the underlying is the stock index.*

Background results. The related literature is vast and we only emphasize the main ideas underlying the approximations/expansions.
In the Gaussian framework (deterministic b and σ), the arithmetic mean has an explicit normal density. Most of the existing approximations focus on the computation of the law of the mean of exponentials in this Gaussian case. We notice the moment matching work [TW91] which approximates this sum of correlated log-normal variables by a log-normal one which matches its first and second moments. When the test function Φ is convex (e.g. $x \mapsto x^+$), tight lower and upper bounds have been computed explicitly in [RS95] and [CD05]. In [GY93], the Laplace transform of the time-average of the geometric Brownian motion is made explicit and by a numerical

inversion, one may evaluate (0.4). When the averaging parameter is the time, computing (0.4) is related to solving a PDE in dimension two, or one in some cases, see [RS95] and [DL05]. Coupling semi-analytical approximations and PDE is proposed in [Zha01]. Alternatively, evaluating (0.4) can be performed by Monte Carlo simulations [LT01]: in the case of the mean of exponentials, it is very efficient to use the geometric mean as a control variate, see [KV90].
In a more general framework (b and σ depending also on x), we notice the Markov projection techniques: the average has the same law as the marginal law of a SDE which coefficients can be approximated, for short time, by an explicit expression involving the different coefficients in the space $\mathcal{A}$, see [ABOBF02]. Yoshida derives in [Yos92] an asymptotic expansion for the density of the time-average as the diffusion coefficients are small, using the Watanabe expansion approach; see [KT01] for applications in interest rate pricing. Recently in [FPP11], Foschia *etal.* use the parametrix method to derive approximations in the case of time-average, as the time is small.

Our contribution. As a comparison with these existing works, our contribution is threefold. First, we consider averages w.r.t. a general parameter, handling at once the case of time-average and other averages, and the diffusion family $(X^\alpha)_{\alpha\in\mathcal{A}}$ is general as well. Second, we choose a different point of view for the expansion (see Subsection 2.1), which makes the approximation exact (at least for $\nu(X^\alpha_T)$) if the coefficients do not depend on x . Finally, we provide tractable non-asymptotic error estimates, under various regularity assumptions on φ and Φ: indeed, it is known (see the discussion in [BGM11, Section 2]) that an asymptotic analysis w.r.t. a given parameter may be misleading since one neglects the influence of the other ones while their roles may be equally important.

Organization of the paper. In the next section, we define the notations and assumptions used throughout our work, and we state some preliminary estimates. Main results are given in Section 2: we first expose our approximation methodology, then we state our expansion results (Theorems 2.1 and 2.2), and finally we present few numerical experiments. The proofs in the case of smooth test functions φ and Φ are given in Section 3. Weakening the regularity assumptions of the test functions involves even more technicalities from Malliavin calculus: this is achieved in Sections 4 and 5. Some technical lemmas are given in Appendix.

1 Preliminaries

1.1 Assumptions and notations

- For a vector $y \in \mathbb{R}^k$ $(k \geq 1)$, $|y|$ stands for its Euclidean norm. The scalar product is denoted by $\langle\ ,\ \rangle$.

- For a random variable $Y \in \mathbb{R}^k$ ($k \geq 1$) and for $p \geq 1$, $|Y|_p$ stands for its $\mathbf{L}_p$-norm: $|Y|_p = (\mathbb{E}|Y|^p)^{1/p}$.
- For a family $(y^\alpha)_{\alpha\in\mathcal{A}}$ of vectors of $\mathbb{R}^k$ ($k \geq 1$), we set $|y|_\infty = \sup_{\alpha\in\mathcal{A}} |y^\alpha|$.
- If $f : \mathbb{R} \mapsto \mathbb{R}$ is a smooth function, $f^{(n)}$ denotes its n-th derivative, or simply f', f'' for $n = 1, 2$.

Regarding the coefficients $b^\alpha : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ and $\sigma^\alpha : [0,T] \times \mathbb{R} \mapsto \mathbb{R}^{1\times q}$ (as a row vector) entering in the definition of (0.1), throughout this work we assume

(R) the coefficients b^α and σ^α are measurable in time and four times continuously differentiable in space. In addition, the coefficients b^α, σ^α and their i-th spatial derivatives $b^{\alpha,i} := \partial_x^i b^\alpha$, $\sigma^{\alpha,i} := \partial_x^i \sigma^\alpha$ ($1 \leq i \leq 4$) are uniformly bounded on $[0,T] \times \mathbb{R}$. We set

$$M_1^\alpha := \sup_{1\leq i\leq 4} \sup_{(t,x)\in[0,T]\times\mathbb{R}} (|b^{\alpha,i}(t,x)| + |\sigma^{\alpha,i}(t,x)|),$$
$$M_0^\alpha := \sup_{0\leq i\leq 4} \sup_{(t,x)\in[0,T]\times\mathbb{R}} (|b^{\alpha,i}(t,x)| + |\sigma^{\alpha,i}(t,x)|).$$

To avoid trivial situations, we assume that $(M_0^\alpha)_{\alpha\in\mathcal{A}}$ is not identically 0: $\nu(\{\alpha : M_0^\alpha > 0\}) > 0$. In addition, we assume that the above constants are uniformly bounded w.r.t. α and we set

$$|M_0|_\infty = \sup_{\alpha\in\mathcal{A}} M_0^\alpha, \qquad |M_1|_\infty = \sup_{\alpha\in\mathcal{A}} M_1^\alpha.$$

In particular, coefficients are Lipschitz continuous in space and thus, the strong solutions to the family of SDEs $(X^\alpha)_{\alpha\in\mathcal{A}}$ are simultaneously well defined with probability 1.

As previously mentioned, ν is a probability measure on $\mathcal{A}$: since $\mathcal{A}$ is countable, it is of the form $\nu(\mathrm{d}\alpha) = \sum_{i\geq 1} p_i \delta_{\alpha_i}(\mathrm{d}\alpha)$ (with $p_i \geq 0$, $\alpha_i \in \mathcal{A}$ and $\sum_{i\geq 1} p_i = 1$). Actually, the exact form of the set $\mathcal{A}$, of the α_i's or of the p_i's is unimportant, since all our results are expressed directly in terms of $\nu(\mathrm{d}\alpha)$.

Remark 1.1 *The assumption that $\mathcal{A}$ is countable is not necessary to define $\{X^\alpha : \alpha \in \mathcal{A}\}$ with probability 1. One could take uncountable sets, for instance $\mathcal{A} = \mathbb{R}$, see [Kun84]. But it woud require additional smoothness assumptions on the coefficients w.r.t. α, in order to define a.s. $\{X^\alpha : \alpha \in \mathcal{A}\}$. Regarding the examples we wish to consider (see Example 0.1), these extra smoothness assumptions may be not satisfied, although one may directly define a.s. $\{X^\alpha : \alpha \in \mathcal{A}\}$. That is why we prefer to stick to countable sets $\mathcal{A}$ and then, by a limit argument, to possibly pass to more general cases: our estimates are ready for this extra step since the results are stated only in terms of integrals w.r.t. ν.*

Some of our results rely on a non-degeneracy condition (ellipticity condition):

($\mathbf{E}^\nu$) for some constant $C_E \geq 1$ (the *ellipticity ratio*), one has

$$\frac{T}{C_E}\nu([M_0^{\cdot}]^2) \leq \int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t \leq T\nu([M_0^{\cdot}]^2).$$

Observe that under **($\mathbf{E}^\nu$)**, $\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t > 0$ since $(M_0^\alpha)_{\alpha\in\mathcal{A}}$ is not identically 0. We define similarly **($\mathbf{E}^{\nu_0}$)**where the measure ν_0 is introduced later in 2.6.

Miscellaneous.

- The constant from the Burkholder-Davis-Gundy inequality w.r.t. the $\mathbf{L}_p$-norm is denoted c_p, without any additional explicit reference to this inequality.
- A test function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is exponentially bounded if for two constants $C_\varphi \geq 0$ and $p_\varphi \geq 0$, we have $|\varphi(x)| \leq C_\varphi \exp(p_\varphi |x|)$ for any $x \in \mathbb{R}$.
- A test function $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is polynomially bounded if for two constants $C_\Phi \geq 0$ and $p_\Phi \geq 0$, we have $|\Phi(x)| \leq C_\Phi(1 + |x|^{p_\Phi})$ for any $x \in \mathbb{R}$.
- In the derivation of error estimates, we make use of numerous constants that we simply denote by c as a *generic constant* (which may change from line to line). These constants depend in an increasing way on the model parameters $|M_0|_\infty$ and $|M_1|_\infty$, on the ellipticity ratio C_E, on T, on the index p considered for $\mathbf{L}_p$-norms, on the growth parameters $C_\varphi, p_\varphi, C_\Phi, p_\Phi$ of the test functions, and on other universal constants. The constants c remain bounded as these dependence parameters go to 0. The generic constants do not depend on $(X_0^\alpha)_{\alpha\in\mathcal{A}}$ and ν.
- For two non-negative real numbers x and y, $x \leq_c y$ means that $x \leq cy$ for a generic constant c.

1.2 Preliminary estimates

As a warm up, in order to make the reader familiar with this set-up, we state preliminary estimates (the easy proof is left to the reader).

Lemma 1.1 (Stochastic Fubini-type result) *Consider a family $(f^\alpha : \Omega\times[0,T] \to f_s^\alpha(\omega) \in \mathbb{R}^{1\times q})_{\alpha\in\mathcal{A}}$ of progressively measurable processes. Let $p \geq 2$ and assume that one of the quantities below is finite*

$$\int_{\mathcal{A}} \Big| \int_0^T |f_s^\alpha|^2 \mathrm{d}s \Big|_{p/2}^{1/2} \nu(\mathrm{d}\alpha) \leq T^{1/2} \sup_{\alpha\in\mathcal{A}, 0\leq s\leq T} \Big| f_s^\alpha \Big|_p,$$

then $\int_{\mathcal{A}} \Big(\int_0^T f_s^\alpha \mathrm{d}W_s \Big) \nu(\mathrm{d}\alpha) \in \mathbf{L}_p$ and we can interchange the ν-(discrete) integral and the Itô integral:

$$\int_{\mathcal{A}} \Big(\int_0^T f_s^\alpha \mathrm{d}W_s \Big) \nu(\mathrm{d}\alpha) = \int_0^T \Big(\int_{\mathcal{A}} f_s^\alpha \nu(\mathrm{d}\alpha) \Big) \mathrm{d}W_s.$$

Under similar conditions, the usual Fubini theorem yields $\int_{\mathcal{A}}\Big(\int_0^T f_s^\alpha \mathrm{d}s\Big)\nu(\mathrm{d}\alpha) = \int_0^T \Big(\int_{\mathcal{A}} f_s^\alpha \nu(\mathrm{d}\alpha)\Big)\mathrm{d}s$. The above equalities are repeatedly used in our computations.

The quantities of interest $\nu(X_T^\cdot)$ and $\nu^{\exp}(X_T^\cdot)$ have respectively exponential moments and polynomial moments.

Lemma 1.2 *Under* **(R)***, for any $p \geq 1$, we have*

$$\Big|\exp\Big(\nu(X_T^\cdot)\Big)\Big|_p \leq \exp\Big(\nu(X_0^\cdot)\Big)\exp\Big(T\nu(M_0^\cdot) + \frac{p}{2}T[\nu(M_0^\cdot)]^2\Big),$$
$$\Big|\nu^{\exp}(X_T^\cdot)\Big|_p \leq \nu^{\exp}(X_0^\cdot)\exp\Big(T|M_0|_\infty + \frac{p}{2}T|M_0|_\infty^2\Big).$$

PROOF. From $\int_{\mathcal{A}}\exp(|X_0^\alpha|)\nu(\mathrm{d}\alpha) < \infty$, we easily check that $\nu^{\exp}(X_0^\cdot)$ and $\nu(X_0^\cdot)$ are well-defined and finite. Lemma 1.1 gives $\nu(X_T^\cdot) = \nu(X_0^\cdot) + \int_0^T(\int_{\mathcal{A}} b^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha))\mathrm{d}t + \int_0^T(\int_{\mathcal{A}}\sigma^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha))\mathrm{d}W_t$, which implies

$$\begin{aligned}
&\mathbb{E}(\exp(p\nu(X_T^\cdot))\\
&= \mathbb{E}\Bigg(\exp\Big(p\nu(X_0^\cdot) + \int_0^T\Big[p(\int_{\mathcal{A}} b^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha)) + \frac{p^2}{2}\Big|\int_{\mathcal{A}}\sigma^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha)\Big|^2\Big]\mathrm{d}t\Big)\\
&\qquad \times\exp\Big(\int_0^T(p\int_{\mathcal{A}}\sigma^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha))\mathrm{d}W_t - \frac{p^2}{2}\int_0^T\Big|\int_{\mathcal{A}}\sigma^\alpha(t, X_t^\alpha)\nu(\mathrm{d}\alpha)\Big|^2\mathrm{d}t\Big)\Bigg)\\
&\leq \exp\Big(p\nu(X_0^\cdot)\Big)\exp\Big(pT\nu(M_0^\cdot) + \frac{p^2}{2}T\nu(M_0^\cdot)^2\Big),
\end{aligned}$$

since the expectation of the exponential term at the third line is equal to 1.
To prove the estimate on $\nu^{\exp}(X_T^\cdot)$, apply the Minkowski inequality to obtain

$$\begin{aligned}
\Big|\nu^{\exp}(X_T^\cdot)\Big|_p \leq \int_{\mathcal{A}}\Big|\exp(X_T^\alpha)\Big|_p\nu(\mathrm{d}\alpha) &\leq \int_{\mathcal{A}}\exp\Big(X_0^\alpha + TM_0^\alpha + \frac{p}{2}T[M_0^\alpha]^2\Big)\nu(\mathrm{d}\alpha)\\
&\leq \nu^{\exp}(X_0^\cdot)\exp(T|M_0|_\infty + \frac{p}{2}T|M_0|_\infty^2).
\end{aligned}$$

□

2 Main results

2.1 Discussion about the approximation methodology: proxy and smart parametrization

Now let us discuss informally our strategy to approximate the laws of $\nu(X_T^\cdot)$ and $\nu^{\exp}(X_T^\cdot)$. Full justifications are given in the next sections.

Regarding the *arithmetic mean*, the idea is to use a Gaussian proxy $X_T^{\alpha,P}$ for X_T^α, which is obtained by freezing the space variable in the coefficients $b^\alpha(s, X_s^\alpha)$ and $\sigma^\alpha(s, X_s^\alpha)$ to its initial value X_0^α. It writes

$$X_t^{\alpha,P} = X_0^\alpha + \int_0^t b^\alpha(s, X_0^\alpha)\mathrm{d}s + \int_0^t \sigma^\alpha(s, X_0^\alpha)\mathrm{d}W_s, \tag{2.5}$$

which defines a Gaussian process. The superscript P refers to the label Proxy. This choice has the advantage that $\nu(X_T^{\cdot,P})$ has an explicit Gaussian law (see Proposition 2.2). The approximations $b^\alpha(s, X_s^\alpha) \approx b^\alpha(s, X_0^\alpha)$ and $\sigma^\alpha(s, X_s^\alpha) \approx \sigma^\alpha(s, X_0^\alpha)$ are expected to be accurate in three cases:

(1) if the spatial derivatives of b^α and σ^α are small. In our error estimates, it is encoded into the constants $(M_1^\alpha)_{\alpha\in\mathcal{A}}$.
(2) if the final time T is small (inducing that $X_s^\alpha \approx X_0^\alpha$ for $s \leq T$).
(3) if the coefficients b^α and σ^α are small, implying again $X_s^\alpha \approx X_0^\alpha$ for $s \leq T$. This is encoded into the constants $(M_0^\alpha)_{\alpha\in\mathcal{A}}$.

This proxy approach has been successfully developed in [BGM09][BGM10][BGM11], to approximate expectations of the marginal X_T of a scalar diffusion process.
Actually we do not only replace X_T^α by $X_T^{\alpha,P}$ in $\nu(X_T^\cdot)$, we also provide correction terms in order to achieve a higher accuracy (see Theorems 2.1 and 2.2). The choice of the proxy (and the computation of correction terms) is also related to a suitable parametrization given by

$$X_t^\alpha(\epsilon) := X_0^\alpha + \epsilon\Big(\int_0^t b^\alpha(s, X_s^\alpha(\epsilon))\mathrm{d}s + \int_0^t \sigma^\alpha(s, X_s^\alpha(\epsilon))\mathrm{d}W_s\Big),$$

so that $X_t^\alpha = X_t^\alpha(\epsilon)|_{\epsilon=1}$, while the proxy $X_t^{\alpha,P} = X_t^\alpha(\epsilon)|_{\epsilon=0} + \partial_\epsilon X_t^\alpha(\epsilon)|_{\epsilon=0}$ appears as the first terms of a Taylor expansion (see Subsection 3.1). Note that this parametrization is different from that of small time asymptotics (see [Wat87]) for which the ϵ-factor for the drift would be ϵ^2. It is also different from that of small noise expansion (see [FW98] or [Yos92][KT01]) for which the ϵ-factor for the drift would be 1. A major advantage of our parametrization/expansion is to be exact for space-independent coefficients b and σ.

Regarding the *mean of exponentials* $\nu^{\exp}(X_T^\cdot)$, one could take as a proxy the exponential of an arithmetic mean $\exp(\nu(X_T^\cdot))$ (this is the usual *arithmetic/geometric mean* approximation), and then approximate the arithmetic mean as before. Let us discuss a bit on this first possible step. By the Jensen inequality, we have

$$\exp(\nu(X_T^\cdot)) \leq \nu^{\exp}(X_T^\cdot);$$

this proxy systematically underestimates the mean of exponentials. The inequality becomes an equality if X_T^α does not depend on α. Moreover, one expects the approximation to be accurate if X_T^α does not depend much on α. However, we do not intend to leverage on this kind of asymptotics, because in practical examples, the dependence of X_T^α w.r.t. α is not small (due for instance to X_0^α). Thus we choose another approximation point of view, coherent with the choice of the proxy $X_T^{\alpha,P}$, i.e. small fluctuations of $X_T^\alpha - X_0^\alpha$ (somehow also meaning a low dependence w.r.t. α). This justifies the introduction of a new probability measure $\nu_0(\mathrm{d}\alpha)$ which takes into account X_0^α:

$$\nu_0(\mathrm{d}\alpha) = \frac{\exp(X_0^\alpha)}{\nu^{\exp}(X_0^{\cdot})}\nu(\mathrm{d}\alpha). \tag{2.6}$$

Then, the mean of exponentials (0.3) writes

$$\nu^{\exp}(X_T^{\cdot}) = \nu^{\exp}(X_0^{\cdot}) \int_{\mathcal{A}} \exp(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha) = \nu^{\exp}(X_0^{\cdot})\nu_0^{\exp}(X_T^{\cdot} - X_0^{\cdot}),$$

and the proxy is given by the exponential of the arithmetic mean, that is

$$\nu^{\exp,P}(X_T^{\cdot}) := \nu^{\exp}(X_0^{\cdot}) \exp\Big(\nu_0(X_T^{\cdot} - X_0^{\cdot})\Big). \tag{2.7}$$

The connection between this latter quantity and $\nu^{\exp}(X_T^{\cdot})$ is made via the smart parametrization ($\epsilon \in [0,1]$)

$$I(\epsilon) = \nu^{\exp}(X_0^{\cdot}) \int_{\mathcal{A}} \exp\left(\epsilon(X_T^\alpha - X_0^\alpha) + (1-\epsilon)\int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\right)\nu_0(\mathrm{d}\alpha), \tag{2.8}$$

so that $I(1) = \nu^{\exp}(X_T^{\cdot})$ and $I(0) = \nu^{\exp,P}(X_T^{\cdot})$. Clearly, almost surely $\epsilon \mapsto I(\epsilon)$ is smooth and a direct computation yields

$$\begin{aligned} I^{(n)}(\epsilon) = \nu^{\exp}(X_0^{\cdot}) \int_{\mathcal{A}} &\exp\left(\epsilon(X_T^\alpha - X_0^\alpha) + (1-\epsilon)\int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\right) \\ &\times \left((X_T^\alpha - X_0^\alpha) - \int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\right)^n \nu_0(\mathrm{d}\alpha). \end{aligned} \tag{2.9}$$

Observe that $I^{(1)}(0) = 0$ and $I^{(2)}(\epsilon) \geq 0$. Thus, by a Taylor formula, we obtain

$$\nu^{\exp}(X_T^{\cdot}) - \nu^{\exp,P}(X_T^{\cdot}) = I(1) - I(0) = \int_0^1 (1-\epsilon)I^{(2)}(\epsilon)\mathrm{d}\epsilon \geq 0, \tag{2.10}$$

proving again that our proxy underestimates the mean of exponentials. Moreover, this parametrization gives a tractable representation of the distance between the two quantities. We obtain the following result, which states that the exponential mean approximation yields an error of order *two* w.r.t. the amplitude of coefficients and $\sqrt{T}$.

Proposition 2.1 *Under* **(R)**, *for any* $p \geq 1$, *we have*

$$\Big|\nu^{\exp}(X_T^{\cdot}) - \nu^{\exp,P}(X_T^{\cdot})\Big|_p \leq c\ \nu_0([M_0^{\cdot}]^2)\ T\ \nu^{\exp}(X_0^{\cdot}) \leq c\Big(|M_0|_\infty\sqrt{T}\Big)^2\nu^{\exp}(X_0^{\cdot}).$$

PROOF. Starting from (2.10) and applying the Minkowski inequality, we have

$$\Big|\nu^{\exp}(X_T^{\cdot}) - \nu^{\exp,P}(X_T^{\cdot})\Big|_p \leq \int_0^1 (1-\epsilon)|I^{(2)}(\epsilon)|_p \mathrm{d}\epsilon.$$

Then, the proof is complete using the estimate (2.11) below. Namely, starting from (2.9), using the Minkowski, Hölder and Jensen inequalities, and similar upper bounds that those in the proof of Lemma 1.2, we obtain for any $n \geq 1$

$$\begin{aligned}
\frac{\Big|I^{(n)}(\epsilon)\Big|_p}{\nu^{\exp}(X_0^{\cdot})} &\leq \int_{\mathcal{A}} \Big(\Big|\exp(\epsilon(X_T^\alpha - X_0^\alpha))\Big|_{3p} \times \Big|\exp\Big((1-\epsilon)\int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big)\Big|_{3p} \\
&\quad \times \Big|(X_T^\alpha - X_0^\alpha) - \int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big|_{3np}^n\Big)\nu_0(\mathrm{d}\alpha) \\
&\leq \int_{\mathcal{A}} \Big(\Big|\exp(\epsilon(X_T^\alpha - X_0^\alpha))\Big|_{3p} \times \Big(\int_{\mathcal{A}}\Big|\exp\Big((1-\epsilon)(X_T^\alpha - X_0^\alpha)\Big)\Big|_{3p}\nu_0(\mathrm{d}\alpha)\Big) \\
&\quad \times \Big(\Big|X_T^\alpha - X_0^\alpha\Big|_{3np} + \int_{\mathcal{A}}\Big|X_T^\alpha - X_0^\alpha\Big|_{3np}\nu_0(\mathrm{d}\alpha)\Big)^n\Big)\nu_0(\mathrm{d}\alpha) \\
&\leq \int_{\mathcal{A}} \Big(e^{T|M_0|_\infty + \frac{3}{2}pT|M_0|_\infty^2} \times e^{T|M_0|_\infty + \frac{3}{2}pT|M_0|_\infty^2} \\
&\quad \times \Big((M_0^\alpha T + c_{3np}M_0^\alpha\sqrt{T}) + \int_{\mathcal{A}}(M_0^\alpha T + c_{3np}M_0^\alpha\sqrt{T})\nu_0(\mathrm{d}\alpha)\Big)^n\Big)\nu_0(\mathrm{d}\alpha) \\
&\leq 2^n e^{2T|M_0|_\infty + 3pT|M_0|_\infty^2}(T + c_{3np}\sqrt{T})^n \nu_0\Big([M_0^{\cdot}]^n\Big). \qquad (2.11)
\end{aligned}$$

□

Next to this first step, we combine the previous approximation of X_T^α, so that the final proxy for $\nu^{\exp}(X_T^{\cdot})$ is given by (instead of (2.7))

$$\nu^{\exp,P}(X_T^{\cdot,P}) := \nu^{\exp}(X_0^{\cdot})\exp\Big(\nu_0(X_T^{\cdot,P} - X_0^{\cdot})\Big). \qquad (2.12)$$

The latter random variable is log-normal, with explicit characteristics, see Proposition 2.3. In the following, we provide correction terms to this approximation, in order to improve the accuracy.

2.2 Weak approximation results

The expansion coefficients are defined through the drift and diffusion coefficients, and their derivatives: we denote them by

$$\begin{aligned}
b_t^\alpha &:= b^\alpha(t, X_0^\alpha), & \sigma_t^\alpha &:= \sigma^\alpha(t, X_0^\alpha), \\
b_t^{\alpha,i} &:= \partial_x^i b^\alpha(t, X_0^\alpha), & \sigma_t^{\alpha,i} &:= \partial_x^i \sigma^\alpha(t, X_0^\alpha), \quad i \geq 1.
\end{aligned} \qquad (2.13)$$

We first state two propositions that give a full validity to the terms arising in our expansion results. The proof is easy and we skip it.

Proposition 2.2 *Under* $(\mathbf{E}^\nu)$, $\nu(X_T^{\cdot,P}) = \int_{\mathcal{A}} X_T^{\alpha,P}\nu(\mathrm{d}\alpha)$ *is distributed as a non-degenerate normal random variable: its mean is equal to* $\int_{\mathcal{A}}(X_0^\alpha + \int_0^T b_t^\alpha \mathrm{d}t)\nu(\mathrm{d}\alpha)$ *and its variance is equal to* $\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha\ \nu(\mathrm{d}\alpha)|^2\mathrm{d}t > 0$. *Then, for any exponentially bounded function* φ, *the mapping* $\epsilon \mapsto \mathbb{E}\Big(\varphi(\nu(X_T^{\cdot,P}) + \epsilon)\Big)$ *is infinitely smooth.*

Proposition 2.3 *Under* $(\mathbf{E}^{\nu_0})$, $\nu^{\exp,P}(X_T^{\cdot,P})$ *is distributed as a non-degenerate log-normal random variable: the mean of* $\ln(\nu^{\exp,P}(X_T^{\cdot,P}))$ *is equal to* $\ln(\nu^{\exp}(X_0^\cdot)) + \int_{\mathcal{A}}\int_0^T b_t^\alpha \mathrm{d}t\nu_0(\mathrm{d}\alpha)$ *and its variance is equal to* $\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha\ \nu_0(\mathrm{d}\alpha)|^2\mathrm{d}t > 0$. *Then, for any polynomially bounded function* Φ, *the mapping* $\epsilon \mapsto \mathbb{E}\Big(\Phi(\nu^{\exp}(X_0^\cdot)\exp(\nu_0(X_T^{\cdot,P} - X_0^\cdot)) + \epsilon)\Big)$ *is infinitely smooth.*

Note that the derivatives exist and are well defined, whatever the smoothness of φ and Φ is. However, the derivatives may exist also without $(\mathbf{E}^\nu)$ or $(\mathbf{E}^{\nu_0})$, provided that φ and Φ are appropriately smooth.

Theorem 2.1 (Expansion formula for the arithmetic mean)
Under $(\mathbf{E}^\nu)$ *and* $(\mathbf{R})$, *for any exponentially bounded function* φ, *define the approximation*

$$\mathbb{E}\left(\varphi(\nu(X_T^\cdot)\right) = \mathbb{E}\left(\varphi(\nu(X_T^{\cdot,P})\right) + \sum_{i=1}^{3}\beta_i^\nu \partial_\epsilon^i \mathbb{E}\left(\varphi(\nu(X_T^{\cdot,P}) + \epsilon)\right)\Big|_{\epsilon=0} + \mathrm{Error}^{(2.1)}(\varphi,\nu)$$

where

$$\begin{aligned}
\beta_1^\nu &:= \int_{\mathcal{A}}\int_0^T\int_0^t b_t^{\alpha,1} b_s^\alpha \mathrm{d}s\ \mathrm{d}t\ \nu(\mathrm{d}\alpha),\\
\beta_2^\nu &:= \int_{\mathcal{A}}\int_0^T\int_0^t \Big(b_s^\alpha\langle\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha), \sigma_t^{\alpha,1}\rangle + b_t^{\alpha,1}\langle\int_{\mathcal{A}}\sigma_s^\alpha\nu(\mathrm{d}\alpha),\sigma_s^\alpha\rangle\Big)\mathrm{d}s\ \mathrm{d}t\ \nu(\mathrm{d}\alpha),\\
\beta_3^\nu &:= \int_{\mathcal{A}}\int_0^T\int_0^t \langle\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha),\sigma_t^{\alpha,1}\rangle\langle\int_{\mathcal{A}}\sigma_s^\alpha\nu(\mathrm{d}\alpha),\sigma_s^\alpha\rangle\mathrm{d}s\ \mathrm{d}t\ \nu(\mathrm{d}\alpha).
\end{aligned}$$

Then, the approximation error is estimated as follows.

i) If φ *is a.e. differentiable with exponentially bounded derivative (satisfying* $|\varphi(x)| + |\varphi'(x)| \le C_\varphi \exp(p_\varphi|x|)$*), then*

$$|\mathrm{Error}^{(2.1)}(\varphi,\nu)| \le_c \exp\left(p_\varphi|\nu(X_0^\cdot)|\right)\ \nu(M_1^\cdot[M_0^\cdot]^2)\ T^{3/2}.$$

ii) If φ *is three times continuously differentiable with exponentially bounded derivatives (satisfying* $|\varphi^{(i)}(x)| \le C_\varphi\exp(p_\varphi|x|)$ *for* $i = 0,\dots,3$*), then*

$$|\mathrm{Error}^{(2.1)}(\varphi,\nu)| \le_c \exp\left(p_\varphi|\nu(X_0^\cdot)|\right)\ \nu(M_1^\cdot[M_0^\cdot]^2)\ T^{3/2}.$$

In this case, $(\mathbf{E}^\nu)$ *is not needed.*

Theorem 2.2 (Expansion formula for the mean of exponentials) *Under* $(\mathbf{E}^{\nu_0})$ *and* $(\mathbf{R})$*, for any polynomially bounded function* Φ*, define the approximation*

$$\begin{aligned}\mathbb{E}\Big(\Phi\Big(\nu^{\exp}(X_T^{\cdot})\Big)\Big) =&\mathbb{E}\Big(\Phi\Big(\nu^{\exp}(X_0^{\cdot})\exp(\nu_0(X_T^{\cdot,P}-X_0^{\cdot}))\Big)\Big)\\ &+\sum_{i=1}^{3}(\beta_i^{\nu_0}+\gamma_i)\partial_\epsilon^i\mathbb{E}\Big(\Phi\Big(\nu^{\exp}(X_0^{\cdot})\exp(\nu_0(X_T^{\cdot,P}-X_0^{\cdot})+\epsilon)\Big)\Big)|_{\epsilon=0}\\ &+\mathrm{Error}^{(2.2)}(\Phi,\nu)\end{aligned}$$

where $(\beta_i^{\nu_0})_{1\le i\le 3}$ *are defined in Theorem 2.1 (replacing* ν *by* ν_0*) and*

$$\begin{aligned}\gamma_1 &:= \frac{1}{2}\int_{\mathcal{A}}\Big(\int_0^T(b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}t\Big)^2\nu_0(\mathrm{d}\alpha)+\frac{1}{2}\int_{\mathcal{A}}\int_0^T\Big|\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\Big|^2\mathrm{d}t\ \nu_0(\mathrm{d}\alpha),\\ \gamma_2 &:= \int_{\mathcal{A}}\Big(\int_0^T\langle\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\mathrm{d}t\Big)\Big(\int_0^T(b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}t\Big)\nu_0(\mathrm{d}\alpha),\\ \gamma_3 &:= \frac{1}{2}\int_{\mathcal{A}}\Big(\int_0^T\langle\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\mathrm{d}t\Big)^2\nu_0(\mathrm{d}\alpha).\end{aligned}$$

Then, the approximation error is estimated as follows.

i) If Φ *is a.e. differentiable with polynomially bounded derivative (satisfying* $|\Phi(x)|\le C_\Phi(1+|x|^{p_\Phi})$ *and* $|\Phi'(x)|\le C_\Phi(1+|x|^{p_\Phi-1})$ *for some* $p_\Phi\ge 1$*), then*

$$|\mathrm{Error}^{(2.2)}(\Phi,\nu)|\le_c \exp(p_\Phi|\nu_0(X_0^{\cdot})|)\ \nu_0([M_0^{\cdot}]^3)\ T^{3/2}.$$

ii) If Φ *is three times continuously differentiable with polynomially bounded derivatives (satisfying* $|\Phi^{(i)}(x)|\le C_\Phi(1+|x|^{p_\Phi-i})$ *for* $i=0,\dots,3$ *for some* $p_\Phi\ge 3$*), then*

$$|\mathrm{Error}^{(2.2)}(\Phi,\nu)|\le_c \exp(p_\Phi|\nu_0(X_0^{\cdot})|)\ \nu_0([M_0^{\cdot}]^3)\ T^{3/2}.$$

In this case, $(\mathbf{E}^{\nu_0})$ *is not needed.*

These results state that for a large class of test functions φ and Φ, the approximation error is of order three w.r.t. the coefficients and $\sqrt{T}$ (suitably averaged w.r.t. ν or ν_0). Observe that the expansion involves only scalar normal or log-normal random variables: it allows simplified numerical evaluations (sometimes in closed forms).
It is a useful property that the expansion coefficients $(\beta_i^\nu,\gamma_i)_{1\le i\le 3}$ do not depend on the functions φ and Φ. It allows more efficient computations, when several functions φ and Φ have to be considered at the same time. Besides, it is possible to derive iterative formulas for computing the coefficients for several times T, similarly to [BGM09, Proposition 4.1], which may be an additional gain in numerical efficiency.

Theorems 2.1 and 2.2 are proved in Section 3 for the smooth case. The case *i)* is more technical and requires intricate Malliavin calculus estimates; it is postponed to Section 4 for Theorem 2.1 and Section 5 for Theorem 2.2. For test functions with less regularity, see Remark 4.1.

2.3 Computation of coefficients in some examples

For the convenience of the reader, we provide the expressions of the expansion coefficients in some specific cases.

(1) *Time-independent coefficients.* Assume that $b^\alpha(s,x) = b^\alpha(x)$ and $\sigma^\alpha(s,x) = \sigma^\alpha(x)$ and set $\bar\sigma^\nu = \nu(\sigma^\cdot(X_0^\cdot))$, $\bar\sigma^{\nu_0} = \nu_0(\sigma^\cdot(X_0^\cdot))$ and $\bar b^{\nu_0} = \nu_0(b^\cdot(X_0^\cdot))$. Then, the time-integrals simplify much and we obtain

$$
\begin{aligned}
\beta_1^\nu &= \frac{T^2}{2}\int_{\mathcal{A}} \partial_x^1 b^\alpha(X_0^\alpha) b^\alpha(X_0^\alpha)\nu(\mathrm{d}\alpha),\\
\beta_2^\nu &= \frac{T^2}{2}\int_{\mathcal{A}} \Big(b^\alpha(X_0^\alpha)\langle\bar\sigma^\nu, \partial_x^1\sigma^\alpha(X_0^\alpha)\rangle + \partial_x^1 b^\alpha(X_0^\alpha)\langle\bar\sigma^\nu, \sigma^\alpha(X_0^\alpha)\rangle\Big)\,\nu(\mathrm{d}\alpha),\\
\beta_3^\nu &= \frac{T^2}{2}\int_{\mathcal{A}} \langle\bar\sigma^\nu, \partial_x^1\sigma^\alpha(X_0^\alpha)\rangle\langle\bar\sigma^\nu, \sigma^\alpha(X_0^\alpha)\rangle\nu(\mathrm{d}\alpha),\\
\gamma_1 &= \frac{T^2}{2}\int_{\mathcal{A}} (b^\alpha(X_0^\cdot) - \bar b^{\nu_0})^2\nu_0(\mathrm{d}\alpha) + \frac{T}{2}\int_{\mathcal{A}} \left|\sigma^\alpha(X_0^\alpha) - \bar\sigma^{\nu_0}\right|^2\,\nu_0(\mathrm{d}\alpha),\\
\gamma_2 &= T^2\int_{\mathcal{A}} \langle\bar\sigma^{\nu_0}, \sigma^\alpha(X_0^\alpha) - \bar\sigma^{\nu_0}\rangle(b^\alpha(X_0^\alpha) - \bar b^{\nu_0})\nu_0(\mathrm{d}\alpha),\\
\gamma_3 &= \frac{T^2}{2}\int_{\mathcal{A}} \Big(\langle\bar\sigma^{\nu_0}, \sigma^\alpha(X_0^\alpha) - \bar\sigma^{\nu_0}\rangle\Big)^2\nu_0(\mathrm{d}\alpha).
\end{aligned}
$$

(2) *Time-average.* Here, we consider the case $\mathcal{A} = [0,T]$, $b^\alpha(s,x) = b(x)\mathbf{1}_{s\le\alpha}$ and $\sigma^\alpha(s,x) = \sigma(x)\mathbf{1}_{s\le\alpha}$, ν being the uniform measure on $\mathcal{A}$; see Example 0.1. Then, we get:

$$
\begin{aligned}
\beta_1^\nu &= \frac{T^2}{6} b(X_0)\partial_x^1 b(X_0),\\
\beta_2^\nu &= \frac{T^2}{12} b(X_0)\sigma(X_0)\partial_x^1\sigma(X_0) + \frac{T^2}{8}\sigma^2(X_0)\partial_x^1 b(X_0),\\
\beta_3^\nu &= \frac{T^2}{15}\sigma^3(X_0)\partial_x^1\sigma(X_0),\\
\gamma_1 &= \frac{T^2}{24} b^2(X_0) + \frac{T}{12}\sigma^2(X_0),\\
\gamma_2 &= \frac{T^2}{24}\sigma^2(X_0)b(X_0),\\
\gamma_3 &= \frac{T^2}{90}\sigma^4(X_0).
\end{aligned}
$$

2.4 Numerical experiments

We test two examples in order to illustrate the good accuracy of our approximations. We consider the case of time-averaged diffusion.
In the first test, we take $\mathcal{A} = [0,T]$, $T = 1$, $\nu(\mathrm{d}\alpha) = \mathrm{d}\alpha$ and $X_t^\alpha = X_{\alpha\wedge t}$ where

$X_t = X_0 - \frac{1}{2}\sigma^2 t + \sigma W_t$ with $\exp(X_0) = 100$. The mean of exponentials $\nu^{\exp}(X_T^{\cdot}) = \int_0^1 \exp(X_t)\mathrm{d}t$ is considered and we approximate $\mathbb{E}(\int_0^1 \exp(X_t)\mathrm{d}t - K)^+$: this example is related to the pricing of Asian option. The exact value is unknown and as a benchmark, we use the semi-analytical method by Zhang [Zha01] which is known to be very accurate. In Table 1, we choose various and realistic σ and K. From Theorem 2.2, we expect that the smaller the parameter σ, the higher the accuracy. Actually, we observe a 3-digits accuracy, which is very satisfying.

	$\sigma = 0.05$		$\sigma = 0.1$		$\sigma = 0.2$	
K	PDE	Theorem 2.2	PDE	Theorem 2.2	PDE	Theorem 2.2
95	8.80884	8.80883	8.91185	8.91176	9.99565	9.99453
100	4.30823	4.30822	4.91512	4.91497	6.77735	6.77612
105	0.95838	0.95820	2.07006	2.06973	4.29646	4.29496

Table 1
Approximation of $\mathbb{E}(\int_0^1 \exp(X_t)\mathrm{d}t - K)^+$.

In the second test, we let the SDE coefficients be non constant and we set $\sigma(t,x) = 0.2\exp(-0.2\ln(x/100))$ and $b(t,x) = -\frac{1}{2}\sigma^2(t,x)$; $(\exp(X_t))_{0\le t\le T}$ is known as the CEV model. Although the coefficients and their derivatives are not bounded (**(R)** is not satisfied), we expect that our expansions can be generalized to that model as well. For $\mathcal{A}$ and ν, we take a discrete-time average with 27 equally-spaced dates and ν is the uniform measure. As a benchmark, we use a Monte Carlo method, with 5000000 simulations (the 95%-confidence interval width is indicated in parentheses in Table 2). Here again, the accuracy is very good. Additional comparative tests are left for further works.

Table 2

K	90	95	100	105	110
Theorem 2.2	10.0206	5.3266	1.8848	0.3773	0.0395
Monte Carlo	10.0201	5.3269	1.8859	0.3790	0.0402
	($\pm$0.0057)	($\pm$0.0051)	($\pm$0.0045)	($\pm$0.0042)	($\pm$0.0033)

3 Proofs when φ and Φ are smooth

3.1 Proof of Theorem 2.1, case ii)

It is divided into several steps: 1) smart parametrization and distance to a Gaussian proxy; 2) application to the expansion of $\mathbb{E}(\varphi(\nu(X_T^{\cdot})))$; 3) transformation of the correction terms into sensitivities; 4) error estimates.

Step 1. Smart parametrization. We recall the parametrization used in [BGM09] and [BGM10]

$$X_t^\alpha(\epsilon) = X_0^\alpha + \epsilon\Big(\int_0^t b^\alpha(s, X_s^\alpha(\epsilon))\mathrm{d}s + \int_0^t \sigma^\alpha(s, X_s^\alpha(\epsilon))\mathrm{d}W_s\Big), \tag{3.14}$$

so that $X_t^\alpha(\epsilon)|_{\epsilon=1} = X_t^\alpha$. Under **(R)**, the map $\epsilon \mapsto X_t^\alpha(\epsilon)$ is C^3 almost surely (see [Kun84]). Thus, we can define

$$X_t^{\alpha,i}(\epsilon) := \partial_\epsilon^i X_t^\alpha(\epsilon), \qquad X_t^{\alpha,i} := \partial_\epsilon^i X_t^\alpha(\epsilon)|_{\epsilon=0} = X_t^{\alpha,i}(0).$$

Taking advantage of the notation (2.13), we easily obtain

$$\begin{aligned}
X_t^{\alpha,0} &= X_0^\alpha, & \\
\mathrm{d}X_t^{\alpha,1} &= b_t^\alpha \mathrm{d}t + \sigma_t^\alpha \mathrm{d}W_t, & X_0^{\alpha,1} = 0,\\
\mathrm{d}X_t^{\alpha,2} &= 2X_t^{\alpha,1}(b_t^{\alpha,1}\mathrm{d}t + \sigma_t^{\alpha,1}\mathrm{d}W_t), & X_0^{\alpha,2} = 0.
\end{aligned}$$

Thus, the Gaussian proxy for X^α is obtained by the first order expansion of $X^\alpha(\epsilon)$ at $\epsilon = 0$:

$$X_t^{\alpha,P} = X_t^{\alpha,0} + X_t^{\alpha,1} = X_0^\alpha + \int_0^t b_s^\alpha \mathrm{d}s + \int_0^t \sigma_s^\alpha \mathrm{d}W_s.$$

The Taylor formula allows to represent the distance to the proxy:

$$X_t^\alpha - X_t^{\alpha,P} = \int_0^1 (1-\epsilon) X_t^{\alpha,2}(\epsilon)\mathrm{d}\epsilon = \frac{X_t^{\alpha,2}}{2} + \int_0^1 \frac{(1-\epsilon)^2}{2} X_t^{\alpha,3}(\epsilon)\mathrm{d}\epsilon. \tag{3.15}$$

Step 2. Consequently, for smooth and exponentially bounded functions φ as in Theorem 2.1 *ii)*, a Taylor formula applied to φ at $\nu(X_T^{\cdot,P})$ gives

$$\begin{aligned}
\mathbb{E}\left(\varphi(\nu(X_T^\cdot))\right) &= \mathbb{E}\left(\varphi(\nu(X_T^{\cdot,P}))\right) + \mathbb{E}\left(\varphi'(\nu(X_T^{\cdot,P}))\nu(X_T^\cdot - X_T^{\cdot,P})\right)\\
&\quad + \int_0^1 (1-\lambda)\mathbb{E}\left(\varphi''\Big(\nu(X_T^{\cdot,P} + \lambda(X_T^\cdot - X_T^{\cdot,P}))\Big)\Big[\nu(X_T^\cdot - X_T^{\cdot,P})\Big]^2\right)\mathrm{d}\lambda\\
&= \mathbb{E}\left(\varphi(\nu(X_T^{\cdot,P}))\right) + \mathbb{E}\left(\varphi'(\nu(X_T^{\cdot,P}))\nu\Big(\frac{X_T^{\cdot,2}}{2}\Big)\right) + \mathrm{Error}^{(2.1)}(\varphi,\nu)
\end{aligned}$$

with

$$\begin{aligned}
\mathrm{Error}^{(2.1)}(\varphi,\nu) &= \int_0^1 \frac{(1-\epsilon)^2}{2}\mathbb{E}\left(\varphi'(\nu(X_T^{\cdot,P}))\nu(X_T^{\cdot,3}(\epsilon))\right)\mathrm{d}\epsilon\\
&\quad + \int_0^1 (1-\lambda)\mathbb{E}\left(\varphi''\Big(\nu(X_T^{\cdot,P} + \lambda(X_T^\cdot - X_T^{\cdot,P}))\Big)\Big[\nu(X_T^\cdot - X_T^{\cdot,P})\Big]^2\right)\mathrm{d}\lambda.
\end{aligned} \tag{3.16}$$

Step 3. Correction terms. One has

$$\mathbb{E}\left(\varphi'(\nu(X_T^{\cdot,P}))\nu\Big(\frac{X_T^{\cdot,2}}{2}\Big)\right) = \int_{\mathcal{A}} \mathbb{E}\left(\varphi'(\nu(X_T^{\cdot,P}))\frac{X_T^{\alpha,2}}{2}\right)\nu(\mathrm{d}\alpha).$$

Notice that

$$\begin{aligned}\nu(X_T^{\cdot,P}) &= \int_{\mathcal{A}}(X_0^\alpha + \int_0^T b_t^\alpha \mathrm{d}t)\nu(\mathrm{d}\alpha) + \int_0^T(\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha))\mathrm{d}W_t,\\ \frac{X_T^{\alpha,2}}{2} &= \int_0^T(\int_0^t b_s^\alpha \mathrm{d}s + \sigma_s^\alpha \mathrm{d}W_s)(b_t^{\alpha,1}\mathrm{d}t + \sigma_t^{\alpha,1}\mathrm{d}W_t).\end{aligned}$$

Hence, by applying Lemma A.2, one gets

$$\mathbb{E}\Big(\varphi'(\nu(X_T^{\cdot,P}))\nu\Big(\frac{X_T^{\cdot,2}}{2}\Big)\Big) = \sum_{i=1}^3 \beta_i^\nu \partial_\epsilon^i \mathbb{E}\Big(\varphi(\nu(X_T^{\cdot,P}) + \epsilon)\Big)|_{\epsilon=0},$$

where the coefficients $(\beta_i^\nu)_{1\le i\le 3}$ are defined in the statement of Theorem 2.1. Observe that the above derivatives on the right hand side are well defined under the smoothness assumptions on φ, without $(\mathbf{E}^\nu)$.

Step 4. Error estimates. The following estimates hold for any $p \ge 1$:

$$\left\{\begin{aligned}&|X_t^{\alpha,1}(\epsilon)|_p \le_c M_0^\alpha\sqrt{T},\\ &|X_t^{\alpha,2}(\epsilon)|_p \le_c [M_1^\alpha\sqrt{T}][M_0^\alpha\sqrt{T}],\\ &|X_t^{\alpha,3}(\epsilon)|_p \le_c [M_1^\alpha\sqrt{T}][M_0^\alpha\sqrt{T}]^2,\end{aligned}\right. \tag{3.17}$$

uniformly in $t \in [0,T]$ and $\epsilon \in [0,1]$, where the generic constant c does not depend on α. These estimates are proved in [BGM09, p.578-579] when the driving Brownian motion is one-dimensional ($q = 1$) and the extension to $q \ge 1$ is straightforward. Thus, using this and (3.15), it readily follows that (for any $p \ge 1$)

$$\begin{aligned}|\int_{\mathcal{A}} X_t^{\alpha,3}(\epsilon)\nu(\mathrm{d}\alpha)|_p &\le \int_{\mathcal{A}}|X_t^{\alpha,3}(\epsilon)|_p\nu(\mathrm{d}\alpha) \le_c T^{3/2}\ \nu(M_1^\cdot[M_0^\cdot]^2),\\ |\int_{\mathcal{A}}[X_T^\alpha - X_T^{\alpha,P}]\nu(\mathrm{d}\alpha)|_p &\le \int_{\mathcal{A}}\int_0^1(1-\epsilon)|X_t^{\alpha,2}(\epsilon)|_p\mathrm{d}\epsilon\nu(\mathrm{d}\alpha) \le_c T\ \nu(M_1^\cdot M_0^\cdot).\end{aligned}$$

Then, from the error representation (3.16) we easily deduce

$$|\mathrm{Error}^{(2.1)}(\varphi,\nu)| \le_c \exp\Big(p_\varphi|\nu(X_0^\cdot)|\Big) \times \Big(T^{3/2}\nu(M_1^\cdot[M_0^\cdot]^2) + T^2(\nu(M_1^\cdot M_0^\cdot))^2\Big).$$

Applying the Cauchy-Schwarz inequality at the last term, we complete the proof. Note that the assumption $(\mathbf{E}^\nu)$ has not been used in this case of smooth φ. $\square$

3.2 Proof of Theorem 2.2, case ii)

We split the proof into several steps: 1) smart parametrization between the arithmetic and geometric means and application to the expansion of $\mathbb{E}\left(\Phi(\nu^{\exp}(X_T^{\cdot}))\right)$; 2) transformation of the correction terms into sensitivities; 3) error estimates.

Step 1. Apply a second order Taylor expansion to Φ at $\nu^{\exp,P}(X_T^{\cdot})$: it writes

$$\begin{aligned}\mathbb{E}\left(\Phi(\nu^{\exp}(X_T^{\cdot}))\right) =&\mathbb{E}\left(\Phi\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\right)\\ &+\mathbb{E}\left(\Phi'\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\Big(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot})\Big)\right)+\mathcal{E}_1(\Phi)\end{aligned}\tag{3.18}$$

where $\mathcal{E}_1(\Phi)$ denotes the expansion error that should be neglected (see later):

$$\mathcal{E}_1(\Phi):=\int_0^1(1-\lambda)\mathbb{E}\Big(\Phi''\Big(\lambda\nu^{\exp}(X_T^{\cdot})+(1-\lambda)\nu^{\exp,P}(X_T^{\cdot})\Big)\Big(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot})\Big)^2\Big)\mathrm{d}\lambda.$$

In view of (2.12), the first term at the r.h.s. of the equality (3.18) can be approximated using the previous results on arithmetic means, applied to the probability measure ν_0 and to the function

$$\varphi(x)=\Phi\Big(\nu^{\exp}(X_0^{\cdot})\exp(x-\nu_0(X_0^{\cdot}))\Big).\tag{3.19}$$

It is a smooth function, exponentially bounded and its derivatives as well. Let us relate the growth parameters $C_\varphi, p_\varphi, C_\Phi, p_\Phi$ and their dependence w.r.t. $(X_0^\alpha)_{\alpha\in\mathcal{A}}$ and ν. By the Jensen inequality, we obtain

$$0\le c_0:=\nu^{\exp}(X_0^{\cdot})\exp(-\nu_0(X_0^{\cdot}))\le\nu^{\exp}(X_0^{\cdot})\int_{\mathcal{A}}\exp(-X_0^\alpha)\nu_0(\mathrm{d}\alpha)=1.\tag{3.20}$$

It enables us to write $|\varphi(x)|\le C_\Phi(1+c_0^{p_\Phi}e^{p_\Phi x})\le 2C_\Phi e^{p_\Phi|x|}$. Similarly, we have $|\varphi^{(1)}(x)|\le c_0e^xC_\Phi(1+[c_0e^x]^{p_\Phi-1})\le 2C_\Phi e^{p_\Phi|x|}$, $|\varphi^{(2)}(x)|\le 4C_\Phi e^{p_\Phi|x|}$ and $|\varphi^{(3)}(x)|\le 10C_\Phi e^{p_\Phi|x|}$. Thus, one can take $p_\varphi=p_\Phi$ and $C_\varphi=10C_\Phi$: note that both parameters p_φ and C_φ do not depend on $(X_0^\alpha)_{\alpha\in\mathcal{A}}$ and ν (which is expected when generic constants come into play). In view of (2.7) and Theorem 2.1 in the smooth case, we have

$$\begin{aligned}\mathbb{E}\Big(\Phi\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\Big)&=\mathbb{E}\Big(\varphi\Big(\nu_0(X_T^{\cdot})\Big)\Big)\\&=\mathbb{E}\Big(\varphi\Big(\nu_0(X_T^{\cdot,P})\Big)\Big)+\sum_{i=1}^3\beta_i^{\nu_0}\partial_\epsilon^i\mathbb{E}\Big(\varphi(\nu_0(X_T^{\cdot,P})+\epsilon)\Big)\Big|_{\epsilon=0}+\mathrm{Error}^{(2.1)}\Big(\varphi,\nu_0\Big),\end{aligned}$$

where

$$|\mathrm{Error}^{(2.1)}(\varphi,\nu_0)|\le_c\ \exp\Big(p_\Phi|\nu_0(X_0^{\cdot})|\Big)\ \nu_0(M_1^{\cdot}[M_0^{\cdot}]^2)\ T^{3/2}.\tag{3.21}$$

Notice that c is a new generic constant (that does not depend on ν and $(X_0^\alpha)_{\alpha\in\mathcal{A}}$).

We now handle the second term at the r.h.s. of the equality (3.18). Using the parametrization (2.8) as in (2.10), we have

$$\nu^{\exp}(X_T^{\cdot}) - \nu^{\exp,P}(X_T^{\cdot}) = \frac{I^{(2)}(0)}{2} + \int_0^1 \frac{(1-\epsilon)^2}{2} I^{(3)}(\epsilon)\mathrm{d}\epsilon. \tag{3.22}$$

From the explicit form of $I^{(2)}(0)$ given in (2.9), we write that $\Phi'\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)I^{(2)}(0)$ is equal to

$$\begin{aligned}
&\Phi'\Big(\nu^{\exp}(X_0^{\cdot})\exp\Big(\nu_0(X_T^{\cdot}-X_0^{\cdot})\Big)\Big)\ \nu^{\exp}(X_0^{\cdot})\exp\Big(\nu_0(X_T^{\cdot}-X_0^{\cdot})\Big) \\
&\qquad\qquad\qquad \times \int_{\mathcal{A}} \Big[(X_T^{\alpha}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2 \nu_0(\mathrm{d}\alpha) \\
&= \varphi'\Big(\nu_0(X_T^{\cdot})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha).
\end{aligned}$$

Bringing together these equalities readily leads to

$$\begin{aligned}
&\mathbb{E}\Big(\Phi'\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\Big(\nu^{\exp}(X_T^{\cdot}) - \nu^{\exp,P}(X_T^{\cdot})\Big)\Big) \\
=&\frac{1}{2}\mathbb{E}\Big(\varphi'\Big(\nu_0(X_T^{\cdot})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big) \\
&+\mathbb{E}\Big(\Phi'\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\int_0^1 \frac{(1-\epsilon)^2}{2} I^{(3)}(\epsilon)\mathrm{d}\epsilon\Big) \\
=&\frac{1}{2}\mathbb{E}\Big(\varphi'\Big(\nu_0(X_T^{\cdot,P})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha,P}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha,P}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big) + \mathcal{E}_2(\Phi)
\end{aligned}$$

where the error term $\mathcal{E}_2(\Phi)$ is given by:

$$\begin{aligned}
\mathcal{E}_2(\Phi) = &\ \mathbb{E}\Big(\Phi'\Big(\nu^{\exp,P}(X_T^{\cdot})\Big)\int_0^1 \frac{(1-\epsilon)^2}{2} I^{(3)}(\epsilon)\mathrm{d}\epsilon\Big) \\
&+\frac{1}{2}\mathbb{E}\Big(\varphi'\Big(\nu_0(X_T^{\cdot})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big) \\
&-\frac{1}{2}\mathbb{E}\Big(\varphi'\Big(\nu_0(X_T^{\cdot,P})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha,P}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha,P}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big).
\end{aligned}$$

To summarize, we have proved that

$$\begin{aligned}
&\mathbb{E}\Big(\Phi(\nu^{\exp}(X_T^{\cdot}))\Big) \\
=&\mathbb{E}\Big(\varphi\Big(\nu_0(X_T^{\cdot,P})\Big)\Big) + \sum_{i=1}^{3}\beta_i^{\nu_0}\partial_\epsilon^i\mathbb{E}\Big(\varphi(\nu_0(X_T^{\cdot,P})+\epsilon)\Big)\Big|_{\epsilon=0} \\
&+\frac{1}{2}\mathbb{E}\Big(\varphi'\Big(\nu_0(X_T^{\cdot,P})\Big)\int_{\mathcal{A}} \Big[(X_T^{\alpha,P}-X_0^{\alpha}) - \int_{\mathcal{A}}(X_T^{\alpha,P}-X_0^{\alpha})\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big) \\
&+\mathrm{Error}^{(2.2)}(\Phi,\nu)
\end{aligned}$$

where $\mathrm{Error}^{(2.2)}(\Phi,\nu) = \mathrm{Error}^{(2.1)}(\varphi,\nu_0) + \mathcal{E}_1(\Phi) + \mathcal{E}_2(\Phi)$.

Step 2. In view of the previous decomposition, since the $\beta_i^{\nu_0}$-weights are given explicitly in Theorem 2.1, it only remains to compute the additional correction term (with the $\frac{1}{2}$-factor) coming from the arithmetic/geometric mean approximation. Notice that

$$\begin{aligned}
&\frac{1}{2}\Big[(X_T^{\alpha,P}-X_0^\alpha)-\int_{\mathcal{A}}(X_T^{\alpha,P}-X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big]^2\\
=&\frac{1}{2}\Big[\int_0^T(\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}W_t+(b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}t\Big]^2\\
=&\int_0^T\Big[\int_0^t(\sigma_s^\alpha-\int_{\mathcal{A}}\sigma_s^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}W_s+(b_s^\alpha-\int_{\mathcal{A}}b_s^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}s\Big]\\
&\qquad\times\Big[(\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}W_t+(b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}t\Big]\\
&+\frac{1}{2}\int_0^T\langle\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\mathrm{d}t.
\end{aligned}$$

Hence, using Lemma A.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\varphi'\Big(\nu_0(X_T^{\cdot,P})\Big)\int_{\mathcal{A}}\Big[(X_T^{\alpha,P}-X_0^\alpha)-\int_{\mathcal{A}}(X_T^{\alpha,P}-X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big]^2\ \nu_0(\mathrm{d}\alpha)\Big]\\
&=\sum_{i=1}^3\gamma_i\partial_\epsilon^i\mathbb{E}\Big(\varphi(\nu_0(X_T^{\cdot,P})+\epsilon)\Big)\Big|_{\epsilon=0}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1=&\int_{\mathcal{A}}\int_0^T\int_0^t(b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))(b_s^\alpha-\int_{\mathcal{A}}b_s^\alpha\nu_0(\mathrm{d}\alpha))\mathrm{d}s\mathrm{d}t\nu_0(\mathrm{d}\alpha)\\
&+\frac{1}{2}\int_{\mathcal{A}}\int_0^T\langle\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\mathrm{d}t\nu_0(\mathrm{d}\alpha),\\
\gamma_2=&\int_{\mathcal{A}}\int_0^T\int_0^t\Big((b_s^\alpha-\int_{\mathcal{A}}b_s^\alpha\nu_0(\mathrm{d}\alpha))\langle\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\Big)\mathrm{d}s\mathrm{d}t\nu_0(\mathrm{d}\alpha)\\
&+\int_{\mathcal{A}}\int_0^T\int_0^t\Big((b_t^\alpha-\int_{\mathcal{A}}b_t^\alpha\nu_0(\mathrm{d}\alpha))\langle\int_{\mathcal{A}}\sigma_s^\alpha\nu_0(\mathrm{d}\alpha),\sigma_s^\alpha-\int_{\mathcal{A}}\sigma_s^\alpha\nu_0(\mathrm{d}\alpha)\rangle\Big)\mathrm{d}s\mathrm{d}t\nu_0(\mathrm{d}\alpha),\\
\gamma_3=&\int_{\mathcal{A}}\int_0^T\int_0^t\langle\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha),\sigma_t^\alpha-\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)\rangle\\
&\qquad\times\langle\int_{\mathcal{A}}\sigma_s^\alpha\nu_0(\mathrm{d}\alpha),\sigma_s^\alpha-\int_{\mathcal{A}}\sigma_s^\alpha\nu_0(\mathrm{d}\alpha)\rangle\mathrm{d}s\mathrm{d}t\nu_0(\mathrm{d}\alpha).
\end{aligned}$$

Some easy symmetry-based arguments in the time-integrals give simplifications and lead to the announced expressions of $(\gamma_i)_i$.

Step 3. We now provide error estimates on $\mathrm{Error}^{(2.2)}(\Phi,\nu)=\mathrm{Error}^{(2.1)}(\varphi,\nu_0)+\mathcal{E}_1(\Phi)+\mathcal{E}_2(\Phi)$. The first error contribution has been estimated in (3.21). To handle $\mathcal{E}_1(\Phi)$, we write

$$|\mathcal{E}_1(\Phi)|\le\sup_{\lambda\in[0,1]}|\Phi''(\lambda\nu^{\exp}(X_T^\cdot)+(1-\lambda)\nu^{\exp,P}(X_T^\cdot))|_2\ |\nu^{\exp}(X_T^\cdot)-\nu^{\exp,P}(X_T^\cdot)|_4^2.$$

Since Φ'' is polynomially bounded, we obtain $|\Phi''(\lambda\nu^{\exp}(X_T^\cdot)+(1-\lambda)\nu^{\exp,P}(X_T^\cdot))|_2\le C_\Phi(1+||\nu^{\exp}(X_T^\cdot)|^{p_\Phi-2}|_2+||\nu^{\exp,P}(X_T^\cdot)|^{p_\Phi-2}|_2)$. Since $p_\Phi\ge3$ and thanks to Proposition 2.1 and Lemma 1.2, we derive that $||\nu^{\exp}(X_T^\cdot)|^{p_\Phi-2}|_2 + ||\nu^{\exp,P}(X_T^\cdot)|^{p_\Phi-2}|_2 \leq_c \Big(\nu^{\exp}(X_0^\cdot)\Big)^{p_\Phi-2}$. On the other hand, owing to Proposition 2.1, we have $|\nu^{\exp}(X_T^\cdot) - \nu^{\exp,P}(X_T^\cdot)|_4 \leq_c \nu_0\Big([M_0^\cdot]^2\Big)T\nu^{\exp}(X_0^\cdot)$. Finally, combining these estimates with (3.20), we have proved

$$|\mathcal{E}_1(\Phi)| \leq_c \Big(1 + (\nu^{\exp}(X_0^\cdot))^{p_\Phi-2}\Big)\Big(\nu^{\exp}(X_0^\cdot)\Big)^2\Big(\nu_0([M_0^\cdot]^2)T\Big)^2 \leq_c e^{p_\Phi|\nu_0(X_0^\cdot)|}\Big(\nu_0([M_0^\cdot]^2)T\Big)^2.$$

Now we handle $\mathcal{E}_2(\Phi)$. Its first term is analysed similarly and after some computations (using in particular (2.11)) we obtain

$$\begin{aligned}|\mathbb{E}\Big(\Phi'\Big(\nu^{\exp,P}(X_T^\cdot)\Big)\int_0^1 \frac{(1-\epsilon)^2}{2} I^{(3)}(\epsilon)\mathrm{d}\epsilon\Big)| &\leq_c \Big(1 + (\nu^{\exp}(X_0^\cdot))^{p_\Phi-1}\Big)\nu^{\exp}(X_0^\cdot)\nu_0([M_0^\cdot]^3)T^{\frac{3}{2}} \\ &\leq_c e^{p_\Phi|\nu_0(X_0^\cdot)|}\nu_0([M_0^\cdot]^3)T^{\frac{3}{2}}.\end{aligned}$$

It remains to upper bound the difference

$$\begin{aligned}&\Big|\frac{1}{2}\mathbb{E}\left(\varphi'\Big(\nu_0(X_T^\cdot)\Big)\int_{\mathcal{A}}\Big[(X_T^\alpha - X_0^\alpha) - \int_{\mathcal{A}}(X_T^\alpha - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big]^2\nu_0(\mathrm{d}\alpha)\right) \\ &\quad - \frac{1}{2}\mathbb{E}\left(\varphi'\Big(\nu_0(X_T^{\cdot,P})\Big)\int_{\mathcal{A}}\Big[(X_T^{\alpha,P} - X_0^\alpha) - \int_{\mathcal{A}}(X_T^{\alpha,P} - X_0^\alpha)\nu_0(\mathrm{d}\alpha)\Big]^2\nu_0(\mathrm{d}\alpha)\right)\Big| \\ &\quad \leq_c e^{p_\Phi|\nu_0(X_0^\cdot)|}\nu_0(M_1^\cdot M_0^\cdot)\nu_0([M_0^\cdot]^2)T^2 + e^{p_\Phi|\nu_0(X_0^\cdot)|}\nu_0(M_1^\cdot M_0^\cdot)\nu_0(M_0^\cdot)T^{3/2},\end{aligned}$$

using similar arguments as before. We skip details. Gathering the previous estimates gives the announced error bound.
Notice that $(\mathbf{E}^{\nu_0})$ is not required in this error analysis. □

4 Proof of Theorem 2.1 for a differentiable function φ (case *i)*)

4.1 Malliavin calculus estimates

To compensate the lack of smoothness of φ, we take advantage of some smoothness results of the SDEs under non-degeneracy condition, using Malliavin calculus techniques. For the related theory and notations for the k-th Malliavin derivatives $\mathcal{D}^k$ and the spaces $\mathbf{D}^{k,p}$, we refer to the book of Nualart [Nua06].

First, we extend (3.17) to Malliavin derivatives. Under **(R)**, for any $\epsilon \in [0,1]$, $t \in$

$[0,T]$, we have $X_t^\alpha(\epsilon)\in\mathbf{D}^{4,\infty}$, $X_t^{\alpha,1}(\epsilon)\in\mathbf{D}^{3,\infty}$, $X_t^{\alpha,2}(\epsilon)\in\mathbf{D}^{2,\infty}$ and for any $p\geq 1$

$$
\left\{\begin{array}{ll}
|\mathcal{D}_r X_t^\alpha(\epsilon)|_p \leq_c |\sigma^\alpha|_\infty, & |\mathcal{D}_r X_t^{\alpha,P}(\epsilon)|_p \leq_c |\sigma^\alpha|_\infty,\\
|\mathcal{D}^2_{r,s} X_t^\alpha(\epsilon)|_p \leq_c |\sigma^\alpha|_\infty M_1^\alpha, & |\mathcal{D}^3_{r,s,u} X_t^\alpha(\epsilon)|_p \leq_c |\sigma^\alpha|_\infty (M_1^\alpha)^2,\\
|\mathcal{D}_r X_t^{\alpha,1}(\epsilon)|_p \leq_c M_0^\alpha, & |\mathcal{D}^2_{r,s} X_t^{\alpha,1}(\epsilon)|_p \leq_c M_0^\alpha M_1^\alpha,\\
|\mathcal{D}_r X_t^{\alpha,2}(\epsilon)|_p \leq_c M_1^\alpha M_0^\alpha\sqrt{T}, & |\mathcal{D}^2_{r,s} X_t^{\alpha,2}(\epsilon)|_p \leq_c M_0^\alpha M_1^\alpha,\\
|\mathcal{D}_r X_t^{\alpha,3}(\epsilon)|_p \leq_c M_1^\alpha (M_0^\alpha\sqrt{T})^2, &
\end{array}\right. \tag{4.23}
$$

uniformly in $(r,s,t,u)\in[0,T]^4$ and $\epsilon\in[0,1]$. The constant c does not depend on α. These estimates are proved in [BGM09, p.581-582] when W is a one-dimensional Brownian motion but the extension to our case $q\geq 1$ is straightforward.
In addition, $\nu(X_T^{\cdot,P})$ and $\nu(X_T^\cdot)$ belong to $\mathbf{D}^{4,\infty}$ (apply [Nua06, Lemma 1.5.3]); furthermore, owing to Minkowski inequalities, for $1\leq k\leq 4$ and $p\geq 1$ we have $|\nu(X_T^\cdot)|_{\mathbf{D}^{k,p}}\leq c_{k,p}\int_{\mathcal{A}}|X_T^\alpha|_{\mathbf{D}^{k,p}}\nu(\mathrm{d}\alpha)$ and similarly for $\nu(X_T^{\cdot,P})$.

4.2 Function regularization by adding a small noise

We introduce a small noise perturbation, which role is to smooth the function φ and to allow the Malliavin calculus integration by parts formula. For this, we consider an extra independent scalar Brownian motion $\widetilde{W}$ and we set

$$\delta=\nu(M_1^\cdot[M_0^\cdot]^2)T. \tag{4.24}$$

Assume w.l.o.g. $\delta\neq 0$ and define

$$\varphi_\delta(x):=\mathbb{E}(\varphi(x+\delta\widetilde{W}_{2T}))=\mathbb{E}(\varphi_{\delta/\sqrt{2}}(x+\delta\widetilde{W}_T)), \tag{4.25}$$

which is a C^∞-function. The two next lemmas quantify the error induced by taking φ_δ instead of φ in the computation of expectations and related sensitivities.

Lemma 4.1 *We have*

$$
\begin{aligned}
&\Big|\mathbb{E}\Big(\varphi_\delta(\nu(X_T^\cdot))\Big)-\mathbb{E}\Big(\varphi(\nu(X_T^\cdot))\Big)\Big|+\Big|\mathbb{E}\Big(\varphi_\delta(\nu(X_T^{\cdot,P}))\Big)-\mathbb{E}\Big(\varphi(\nu(X_T^{\cdot,P}))\Big)\Big|\\
&\qquad\leq_c \delta\sqrt{T}\exp\Big(p_\varphi|\nu(X_0^\cdot)|\Big)\\
&\qquad\leq_c \nu(M_1^\cdot[M_0^\cdot]^2)T^{3/2}\exp\Big(p_\varphi|\nu(X_0^\cdot)|\Big).
\end{aligned}
$$

PROOF. Observe that $\mathbb{E}(\varphi_\delta(\nu(X_T^\cdot))=\mathbb{E}(\varphi(\nu(X_T^\cdot)+\delta\widetilde{W}_{2T}))$: then, the first inequality follows by using the differentiability property of φ and usual moment estimates, such as those of Lemma 1.2. $\square$

Lemma 4.2 *For any $i \geq 1$, we have*

$$\Big|\partial_\epsilon^i \mathbb{E}\Big(\varphi_\delta(\nu(X_T^{\cdot,P})+\epsilon)\Big)|_{\epsilon=0} - \partial_\epsilon^i \mathbb{E}\Big(\varphi(\nu(X_T^{\cdot,P})+\epsilon)\Big)|_{\epsilon=0}\Big|$$
$$\leq_c \frac{\delta\sqrt{T}}{(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t)^{\frac{i}{2}}} \exp\Big(p_\varphi|\nu(X_0^\cdot)|\Big).$$

PROOF. In view of Proposition 2.2, we have

$$\mathbb{E}(\varphi_\delta(\nu(X_T^{\cdot,P})+\epsilon)) = \int_{\mathbb{R}} \mathbb{E}(\varphi(y+\delta \widetilde{W}_{2T})) \frac{\exp\Big(-\frac{(y-\mathbb{E}(\nu(X_T^{\cdot,P}))-\epsilon)^2}{2\mathrm{Var}(\nu(X_T^{\cdot,P}))}\Big)}{\sqrt{2\pi \mathrm{Var}(\nu(X_T^{\cdot,P}))}} dy,$$

and

$$\partial_\epsilon^i \mathbb{E}\Big(\varphi_\delta(\nu(X_T^{\cdot,P})+\epsilon)\Big)|_{\epsilon=0} - \partial_\epsilon^i \mathbb{E}\Big(\varphi(\nu(X_T^{\cdot,P})+\epsilon)\Big)|_{\epsilon=0}$$
$$= \int_{\mathbb{R}} \mathbb{E}[\varphi(y+\delta \widetilde{W}_{2T}) - \varphi(y)] \partial_\epsilon^i \Big(\frac{\exp\Big(-\frac{(y-\mathbb{E}(\nu(X_T^{\cdot,P}))-\epsilon)^2}{2\mathrm{Var}(\nu(X_T^{\cdot,P}))}\Big)}{\sqrt{2\pi \mathrm{Var}(\nu(X_T^{\cdot,P}))}}\Big)|_{\epsilon=0} dy.$$

Then, we easily complete the proof using the regularity property of φ and standard upper bounds for the derivatives of the Gaussian density. □

As a consequence of Lemmas 4.1 and 4.2 and taking into account the magnitude of the coefficients $(\beta_i^\nu)_{1\leq i\leq 3}$ and the ellipticity condition $(\mathbf{E}^\nu)$, observe that *i)* of Theorem 2.1 is proved if we establish

$$|\mathrm{Error}^{(2.1)}(\varphi_\delta,\nu)| \leq_c \exp\Big(p_\varphi|\nu(X_0^\cdot)|\Big)\ \nu(M_1^\cdot[M_0^\cdot]^2)\ T^{3/2}.$$

4.3 Upper bound on the error expansion for φ_δ

From (3.16), recall that $\mathrm{Error}^{(2.1)}(\varphi_\delta,\nu)$ is equal to

$$\int_0^1 \frac{(1-\epsilon)^2}{2} \mathbb{E}\left(\varphi_\delta'(\nu(X_T^{\cdot,P}))\nu(X_T^{\cdot,3}(\epsilon))\right) \mathrm{d}\epsilon$$
$$+ \int_0^1 (1-\lambda)\mathbb{E}\left(\varphi_\delta''\Big(\nu(X_T^{\cdot,P}+\lambda(X_T^\cdot - X_T^{\cdot,P}))\Big)\Big[\nu(X_T^\cdot - X_T^{\cdot,P})\Big]^2\right) \mathrm{d}\lambda.$$

The first term can be handled as in the smooth case (we mainly need $|\varphi_\delta'(x)| = |\mathbb{E}(\varphi'(x+\delta\widetilde{W}_{2T}))| \leq_c \exp(p_\varphi|x|)$) and it gives the announced upper bound. Now, we estimate the second contribution: thanks to (4.25), the absolute value of the second term is equal to

$$
\begin{aligned}
&\Big|\int_0^1 (1-\lambda)\mathbb{E}\left(\varphi''_{\delta/\sqrt{2}}\Big(\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))+\delta\widetilde{W}_T\Big)\Big[\nu(X_T^{\cdot}-X_T^{\cdot,P})\Big]^2\right)\mathrm{d}\lambda\Big|\\
\leq_c &\sup_{\lambda\in[0,1]}\Big|\varphi'_{\delta/\sqrt{2}}\Big(\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))+\delta\widetilde{W}_T\Big)\Big|_2\frac{\Big|[\nu(X_T^{\cdot}-X_T^{\cdot,P})]^2\Big|_{\mathbf{D}^{1,5/2}}}{(\int_0^T|\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha)|^2\mathrm{d}t)^{1/2}}\\
\leq_c &\exp\Big(p_\varphi|\nu(X_0^{\cdot})|\Big)\frac{\Big(\int_{\mathcal{A}}|X_T^\alpha-X_T^{\alpha,P}|_{\mathbf{D}_{1,5}}\nu(\mathrm{d}\alpha)\Big)^2}{(T\nu([M_0^{\cdot}]^2))^{1/2}}\\
\leq_c &\exp\Big(p_\varphi|\nu(X_0^{\cdot})|\Big)\frac{[\nu([M_1^{\cdot}M_0^{\cdot}])T]^2}{(T\nu([M_0^{\cdot}]^2))^{1/2}}\leq_c\ \exp\Big(p_\varphi|\nu(X_0^{\cdot})|\Big)\ \nu(M_1^{\cdot}[M_0^{\cdot}]^2)\ T^{3/2},
\end{aligned}
$$

where at the first inequality we have applied Lemma 4.3, at the second inequality we have used Hölder/Minkowski inequalities and $(\mathbf{E}^\nu)$, and the third inequality follows from (3.17) and (4.23). To complete the proof, it remains to establish the following lemma.

Lemma 4.3 *For any $Z\in\mathbf{D}^{1,\infty}$ and any $\lambda\in[0,1]$, there exists a random variable Z^λ in any $\mathbf{L}_p$ ($p\geq 1$) such that*

$$
\begin{aligned}
&\mathbb{E}\left(\varphi''_{\delta/\sqrt{2}}\Big(\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))+\delta\widetilde{W}_T\Big)Z\right)\\
&=\mathbb{E}\left(\varphi'_{\delta/\sqrt{2}}\Big(\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))+\delta\widetilde{W}_T\Big)Z^\lambda\right)
\end{aligned}
$$

where $|Z^\lambda|_p\leq_c|Z|_{\mathbf{D}^{1,p+\frac12}}(\int_0^T|\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha)|^2\mathrm{d}t)^{-1/2}$, uniformly in λ.

PROOF. We follow a standard Malliavin calculus routine. The difficulty comes from the possible degeneracy of $\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))$ in spite of $(\mathbf{E}^\nu)$, and actually, this is the *raison d'être* of the small perturbation $\delta\widetilde{W}_T$.
Here the Malliavin calculus computations are made w.r.t. the full Brownian motion $(W,\widetilde{W})$ and the Malliavin derivatives associated to the $\widetilde{W}$-component are denoted with a $\tilde{}$. First, $F^\lambda:=\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))+\delta\widetilde{W}_T\in\mathbf{D}^{4,\infty}$ (hence in $\mathbf{D}^{2,\infty}$). Second, its Malliavin covariance matrix $\gamma_{F^\lambda}=\int_0^T|\mathcal{D}_t\nu(X_T^{\cdot,P}+\lambda(X_T^{\cdot}-X_T^{\cdot,P}))|^2\mathrm{d}t+\delta^2T$ is obviously invertible ($\delta\neq 0$). Hence, the existence of Z^λ follows from [Nua06, Proposition 2.1.4] and by [Nua06, Proposition 1.5.6], we obtain

$$
|Z^\lambda|_p\leq_c|\gamma_{F^\lambda}^{-1}|_{\mathbf{D}^{1,2p(2p+1)}}|(\mathcal{D}F^\lambda,\tilde{\mathcal{D}}F^\lambda)|_{\mathbf{D}^{1,2p(2p+1)}}|Z|_{\mathbf{D}^{1,p+\frac12}}. \tag{4.26}
$$

It remains to estimate the two first $\mathbf{D}^{1,2p(2p+1)}$-norms related to F^λ. First, we have $\tilde{\mathcal{D}}_tF^\lambda=\delta\mathbf{1}_{t\leq T}$ and

$$
\begin{aligned}
\mathcal{D}_tF^\lambda&=\int_{\mathcal{A}}(\lambda\mathcal{D}_tX_T^\alpha+(1-\lambda)\mathcal{D}_tX_T^{\alpha,P})\nu(\mathrm{d}\alpha)\\
&=\mathbf{1}_{t\leq T}\int_{\mathcal{A}}(\lambda\nabla X_T^\alpha(\nabla X_t^\alpha)^{-1}\sigma^\alpha(t,X_t^\alpha)+(1-\lambda)\sigma_t^\alpha)\nu(\mathrm{d}\alpha)\\
&=\mathbf{1}_{t\leq T}(\int_{\mathcal{A}}\sigma_t^\alpha\nu(\mathrm{d}\alpha))+\mathbf{1}_{t\leq T}\int_{\mathcal{A}}\lambda(\nabla X_T^\alpha(\nabla X_t^\alpha)^{-1}\sigma^\alpha(t,X_t^\alpha)-\sigma_t^\alpha)\nu(\mathrm{d}\alpha)
\end{aligned}\tag{4.27}
$$

where $\nabla X_t^\alpha = \partial_{X_0^\alpha} X_t^\alpha$ is the first derivative process (see [Nua06, Section 2.2]). This easily implies

$$|(\mathcal{D}F^\lambda, \tilde{\mathcal{D}}F^\lambda)|_{\mathbf{D}^{1,p}} \leq_c \nu(M_0^\cdot)\sqrt{T} \tag{4.28}$$

for any $p \geq 1$, we skip details. Furthermore, from (4.27) it directly follows $\sup_{t\leq T} |\mathcal{D}_t F^\lambda - \mathbf{1}_{t\leq T}(\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha))|_p \leq_c \nu(M_1^\cdot M_0^\cdot)\sqrt{T}$ for any $p \geq 1$, which implies

$$\Big| \int_0^T |\mathcal{D}_t F^\lambda|^2 \mathrm{d}t - \int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t \Big|_p \leq_c T^{3/2} \nu(M_1^\cdot M_0^\cdot)\nu(M_0^\cdot).$$

Thus, for any given $p \geq 1$ and $q \geq 1$, we deduce

$$\begin{aligned}
|\gamma_{F^\lambda}^{-1}|_p &\leq |\gamma_{F^\lambda}^{-1} \mathbf{1}_{\int_0^T |\mathcal{D}_t F^\lambda|^2 \mathrm{d}t < \frac{1}{2}\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t}|_p + 2\Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1} \\
&\leq (\delta^2 T)^{-1} \Big[\mathbb{P}\Big(\int_0^T |\mathcal{D}_t F^\lambda|^2 \mathrm{d}t < \frac{1}{2}\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)\Big]^{1/p} + 2\Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1} \\
&\leq (\delta^2 T)^{-1} \left[\frac{\Big| \int_0^T |\mathcal{D}_t F^\lambda|^2 \mathrm{d}t - \int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big|_q}{\frac{1}{2}\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t}\right]^{q/p} + 2\Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1} \\
&\leq_c (\nu(M_1^\cdot [M_0^\cdot]^2) T^{3/2})^{-2} \Big[\frac{T^{3/2}\nu(M_1^\cdot M_0^\cdot)\nu(M_0^\cdot)}{\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t}\Big]^{q/p} + \Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1}
\end{aligned}$$

where we have used the Markov inequality at the third line. Then, taking $q = 4p$ and owing to $(\mathbf{E}^\nu)$, we conclude

$$\begin{aligned}
|\gamma_{F^\lambda}^{-1}|_p &\leq_c \Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1} \left[(\nu(M_1^\cdot [M_0^\cdot]^2) T^{3/2})^{-2} \frac{\big[T^{3/2}\nu(M_1^\cdot M_0^\cdot)\nu(M_0^\cdot)\big]^4}{[T\nu([M_0^\cdot]^2)]^3} + 1\right] \\
&\leq_c \Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1}.
\end{aligned}$$

Similar computations yield $|\mathcal{D}\gamma_{F^\lambda}|_p \leq_c \nu(M_1^\cdot M_0^\cdot)\nu(M_0^\cdot)T^{3/2}$ and it leads to

$$\begin{aligned}
|\gamma_{F^\lambda}^{-1}|_{\mathbf{D}^{1,p}} &\leq_c \Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1} + \frac{\nu(M_1^\cdot M_0^\cdot)\nu(M_0^\cdot)T^{3/2}}{\Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^2} \\
&\leq_c \Big(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu(\mathrm{d}\alpha)|^2 \mathrm{d}t\Big)^{-1}
\end{aligned}$$

for any $p \geq 1$. Plugging this upper bound and 4.28 in 4.26, and using $(\mathbf{E}^\nu)$ completes the proof of the upper bound for $|Z^\lambda|_p$. □

Remark 4.1 *The error analysis for* $\mathrm{Error}^{(2.1)}(\varphi, \nu)$ *can be performed very analogously if φ is locally Hölder continuous. Here are the main arguments. The estimates of Lemmas 4.1 and 4.2 can be easily extended. Because φ'_δ is no more an exponentially bounded function, Lemma 4.3 should be restated with two integration by parts formulas (instead of one). To maintain an appropriate accuracy, δ should be chosen* *according to the Hölder exponent of φ. Finally, as in [BGM09, Theorem 5.4], the approximation formula of Theorem 2.1 is expected to be less accurate if φ is less smooth. The extension to arbitrary function φ without assuming any a priori regularity would require, in particular, a generalization of Lemma 4.1 (regarding $\nu(X_T^{\cdot})$) to this setting and it is not clear to us how to achieve it under* $(\mathbf{E}^\nu)$. *However, for functions of bounded variations (such as indicator functions of an interval), a generalization of Lemma 4.1 is possible thanks to [Avi09, Theorem 2.4], but under the additional assumption that the law of $\nu(X_T^{\cdot})$ admits a bounded density (it seems that* $(\mathbf{E}^\nu)$ *does not imply neither the existence of density nor its boundedness).*

5 Proof of Theorem 2.2 for a differentiable function Φ (case *i)*)

The analysis follows the lines of Subsection 3.2 and Section 4. We only give the main arguments and we skip details. Let us introduce a regularized function

$$\Phi_\delta(x) := \mathbb{E}(\Phi(x + \delta \widetilde{W}_{2T})) = \mathbb{E}(\Phi_{\delta/\sqrt{2}}(x + \delta \widetilde{W}_T)),$$

where $\widetilde{W}$ is an extra independent scalar Brownian motion and where

$$\delta = \nu^{\exp}(X_0^{\cdot})\nu_0([M_0^{\cdot}]^3)T \neq 0.$$

Let us also define

$$\varphi_\delta(x) := \Phi_\delta\Big(\nu^{\exp}(X_0^{\cdot})\exp(x - \nu_0(X_0^{\cdot}))\Big),$$

which satisfies $\varphi_\delta(x)| + |\varphi_\delta'(x)| \leq_c C_\Phi e^{p_\Phi|\nu_0(X_0^{\cdot})|}e^{p_\Phi|x-\nu_0(X_0^{\cdot})|}$. We now state natural extensions of Lemma 4.1, 4.2 and 4.3; their proofs are very similar and we leave them to the reader.

Lemma 5.1 *We have*

$$\Big|\mathbb{E}\Big(\Phi_\delta(\nu^{\exp}(X_T^{\cdot}))\Big) - \mathbb{E}\Big(\Phi(\nu^{\exp}(X_T^{\cdot}))\Big)\Big| + \Big|\mathbb{E}\Big(\Phi_\delta(\nu^{\exp,P}(X_T^{\cdot,P}))\Big) - \mathbb{E}\Big(\Phi(\nu^{\exp,P}(X_T^{\cdot,P}))\Big)\Big|$$
$$\leq_c \nu_0([M_0^{\cdot}]^3)T^{3/2}\exp\Big(p_\Phi|\nu_0(X_0^{\cdot})|\Big).$$

Lemma 5.2 *For any $i \geq 1$, we have*

$$\Big|\partial_\epsilon^i\mathbb{E}\Big(\Phi_\delta\Big(\nu^{\exp}(X_0^{\cdot})\exp(\nu_0(X_T^{\cdot,P} - X_0^{\cdot}) + \epsilon)\Big)\Big)|_{\epsilon=0}$$
$$-\partial_\epsilon^i\mathbb{E}\Big(\Phi\Big(\nu^{\exp}(X_0^{\cdot})\exp(\nu_0(X_T^{\cdot,P} - X_0^{\cdot}) + \epsilon)\Big)\Big)|_{\epsilon=0}\Big|$$
$$\leq_c \frac{\nu_0([M_0^{\cdot}]^3)T^{3/2}}{(\int_0^T|\int_{\mathcal{A}}\sigma_t^\alpha\nu_0(\mathrm{d}\alpha)|^2\mathrm{d}t)^{\frac{i}{2}}}\exp\Big(p_\Phi|\nu_0(X_0^{\cdot})|\Big).$$

Lemma 5.3 *For any $Z \in \mathbf{D}^{1,\infty}$ and any $\lambda \in [0,1]$, there exist random variables $Z^{1,\lambda}$*

and $Z^{2,\lambda}$ *in any* $\mathbf{L}_p$ $(p \geq 1)$ *such that*

$$\begin{aligned}
&\mathbb{E}\Big(\Phi''_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp}(X_T^\cdot) + (1-\lambda)\nu^{\exp,P}(X_T^\cdot) + \delta\widetilde{W}_T\Big)Z\Big)\\
&\qquad\qquad = \mathbb{E}\Big(\Phi'_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp}(X_T^\cdot) + (1-\lambda)\nu^{\exp,P}(X_T^\cdot) + \delta\widetilde{W}_T\Big)Z^{1,\lambda}\Big),\\
&\mathbb{E}\Big(\Phi''_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp,P}(X_T^\cdot) + (1-\lambda)\nu^{\exp,P}(X_T^{\cdot,P}) + \delta\widetilde{W}_T\Big)Z\Big)\\
&\qquad\qquad = \mathbb{E}\Big(\Phi'_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp,P}(X_T^\cdot) + (1-\lambda)\nu^{\exp,P}(X_T^{\cdot,P}) + \delta\widetilde{W}_T\Big)Z^{2,\lambda}\Big),
\end{aligned}$$

where $|Z^{1,\lambda}|_p + |Z^{2,\lambda}|_p \leq_c |Z|_{\mathbf{D}^{1,p+\frac{1}{2}}}(\nu^{\exp}(X_0^\cdot))^{-1}(\int_0^T |\int_{\mathcal{A}} \sigma_t^\alpha \nu_0(\mathrm{d}\alpha)|^2\mathrm{d}t)^{-1/2}$*, uniformly in* λ*.*

Then, write

$$\begin{aligned}
(5.29)\qquad \mathbb{E}(\Phi(\nu^{\exp}(X_T^\cdot))) &= \mathbb{E}(\Phi(\nu^{\exp}(X_T^\cdot))) - \mathbb{E}(\Phi_\delta(\nu^{\exp}(X_T^\cdot)))\\
&\quad + \mathbb{E}(\Phi_\delta(\nu^{\exp,P}(X_T^\cdot)))\\
&\quad + \mathbb{E}\left(\Phi'_{\delta/\sqrt{2}}\Big(\nu^{\exp,P}(X_T^\cdot) + \delta\widetilde{W}_T\Big)\Big(\nu^{\exp}(X_T^\cdot) - \nu^{\exp,P}(X_T^\cdot)\Big)\right)\\
\int_0^1 (1-\lambda)\mathbb{E}\Big(\Phi''_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp}(X_T^\cdot) &+ (1-\lambda)\nu^{\exp,P}(X_T^\cdot) + \delta\widetilde{W}_T\Big)\Big(\nu^{\exp}(X_T^\cdot) - \nu^{\exp,P}(X_T^\cdot)\Big)^2\Big)\mathrm{d}\lambda.
\end{aligned}$$

The first term at the r.h.s. of (5.29) can be neglected since thanks to Lemma 5.1, it is smaller than $c\nu_0([M_0^\cdot]^3)T^{3/2}\exp(p_\Phi|\nu_0(X_0^\cdot)|)$.

The second term at the r.h.s. of (5.29) can be expanded using Theorem 2.1: we obtain

$$\begin{aligned}
&\mathbb{E}\left(\Phi_\delta\Big(\nu^{\exp,P}(X_T^\cdot)\Big)\right) = \mathbb{E}\Big(\Phi_\delta\Big(\nu^{\exp}(X_0^\cdot)\exp(\nu_0(X_T^{\cdot,P} - X_0^\cdot))\Big)\Big)\\
&\quad + \sum_{i=1}^3 \beta_i^{\nu_0}\partial_\epsilon^i\mathbb{E}\Big(\Phi_\delta\Big(\nu^{\exp}(X_0^\cdot)\exp(\nu_0(X_T^{\cdot,P} - X_0^\cdot) + \epsilon)\Big)\Big)|_{\epsilon=0} + \mathrm{Error}^{(2.1)}\Big(\varphi_\delta, \nu_0\Big),
\end{aligned}$$

with $|\mathrm{Error}^{(2.1)}\Big(\varphi_\delta,\nu_0\Big)| \leq_c \nu_0(M_1^\cdot[M_0^\cdot]^2)T^{3/2}\exp(p_\Phi|\nu_0(X_0^\cdot)|)$. In addition, in the above decomposition, we can replace Φ_δ by Φ (to obtain the main term and the β^{ν_0}-terms in Theorem 2.2): it yields an extra error smaller than $c\nu_0([M_0^\cdot]^3)T^{3/2}\exp(p_\Phi|\nu_0(X_0^\cdot)|)$ (apply Lemma 5.1 and Lemma 5.2 combined with $(\mathbf{E}^{\nu_0})$).

The fourth term at the r.h.s. of (5.29) is estimated by applying Lemma 5.3: after some computations, we obtain the upper bound

$$\begin{aligned}
ce^{(p_\Phi-1)|\nu_0(X_0^\cdot)|} \times (\nu^{\exp}(X_0^\cdot)\nu_0([M_0^\cdot]^2)T)^2 \times (\nu^{\exp}(X_0^\cdot))^{-1}(\int_0^T | \int_{\mathcal{A}} \sigma_t^\alpha\nu_0(\mathrm{d}\alpha)|^2\mathrm{d}t)^{-1/2}\\
\leq_c \nu_0([M_0^\cdot]^3)T^{3/2}\exp(p_\Phi|\nu_0(X_0^\cdot)|)
\end{aligned}$$

using $(\mathbf{E}^{\nu_0})$ and $\nu^{\exp}(X_0^\cdot) \leq \exp(\nu_0(X_0^\cdot))$.

Finally, it remains to show that the third term at the r.h.s. of (5.29) gives the γ-terms of Theorem 2.2 plus a negligible error (upper bounded by $c\nu_0([M_0^\cdot]^3)T^{3/2}\exp(p_\Phi|\nu_0(X_0^\cdot)|)$).

Decompose this third term as follows:

$$
\begin{aligned}
&\mathbb{E}\left(\Phi'_{\delta/\sqrt{2}}\Big(\nu^{\exp,P}(X_T^{\cdot})+\delta\widetilde{W}_T\Big)\Big(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot})\Big)\right)\\
=&\mathbb{E}\left(\Phi'_{\delta/\sqrt{2}}\Big(\nu^{\exp,P}(X_T^{\cdot,P})+\delta\widetilde{W}_T\Big)\Big(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot})\Big)\right)\\
&+\mathbb{E}\Big(\int_0^1\Phi''_{\delta/\sqrt{2}}\Big(\lambda\nu^{\exp,P}(X_T^{\cdot})+(1-\lambda)\nu^{\exp,P}(X_T^{\cdot,P})+\delta\widetilde{W}_T\Big)\mathrm{d}\lambda\\
&\qquad\times(\nu^{\exp,P}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot,P}))(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot}))\Big).
\end{aligned}
$$

An application of Lemma 5.3 (with similar computations as before) ensures that the contribution with the $\mathrm{d}\lambda$-integral yields a negligible error. The remaining term, which is equal to

$$\mathbb{E}\Big(\Phi'_{\delta}\Big(\nu^{\exp,P}(X_T^{\cdot,P})\Big)\Big(\nu^{\exp}(X_T^{\cdot})-\nu^{\exp,P}(X_T^{\cdot})\Big)\Big),$$

can be expanded as in the proof for smooth Φ (starting from (3.22)): the end of the proof is now straightforward, we skip details. □

A Appendix

We derive integration by parts formulas, that are useful for the explicit calculus of the corrections terms. In the following, the process $c_t:[0,T]\to\mathbb{R}^{1\times q}$ (resp. $a_t, f_t, h_t: [0,T]\to\mathbb{R}^{1\times q}$, $e_t, g_t:[0,T]\to\mathbb{R}$) is square integrable and predictable (resp. square integrable and deterministic) and l is a smooth function, exponentially bounded and its derivatives as well.

The one-dimensional version of the above Lemma is proved in [BGM09, Lemma A.2].

Lemma A.1 *We have*

$$\mathbb{E}[l(\int_0^T a_t\mathrm{d}W_t)\int_0^T c_t\mathrm{d}W_t]=\mathbb{E}[l^{(1)}(\int_0^T a_t\mathrm{d}W_t)\int_0^T\langle a_t,c_t\rangle\mathrm{d}t].$$

In the case of deterministic c, it is equal to $(\int_0^T\langle a_t,c_t\rangle\mathrm{d}t)\partial^1_\epsilon\mathbb{E}[l(\int_0^T a_t\mathrm{d}W_t+\epsilon)]|_{\epsilon=0}$.

PROOF. We invoke arguments from Malliavin calculus, following the notation of Nualart [Nua06]. The process a being deterministic, the Malliavin derivative of $\int_0^T a_t\mathrm{d}W_t$ is equal to $\mathcal{D}_t(\int_0^T a_t\mathrm{d}W_t)=a_t\mathbf{1}_{t\le T}$. Owing to the growth conditions on l, $l(\int_0^T a_t\mathrm{d}W_t)\in\mathbf{D}^{1,\infty}$ and $\mathcal{D}_t[l(\int_0^T a_t\mathrm{d}W_t)]=l^{(1)}(\int_0^T a_t\mathrm{d}W_t)a_t\mathbf{1}_{t\le T}$. In addition, $\int_0^T c_t\mathrm{d}W_t$ can be identified with the Skorohod operator applied to c [Nua06, Proposition 1.3.11]: thus, applying the duality relationship of Malliavin calculus [Nua06, Definition 1.3.1], we obtain the first equality. The next statement is clear. □

Lemma A.2 *We have*

$$\mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t)(\int_0^T [\int_0^t g_s \mathrm{d}s + h_s \mathrm{d}W_s](e_t \mathrm{d}t + f_t \mathrm{d}W_t)] = \sum_{i=0}^{2} \lambda_i \partial_\epsilon^i \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t + \epsilon)]|_{\epsilon=0},$$

where

$$\lambda_0 = \int_0^T \int_0^t e_t g_s \mathrm{d}s\mathrm{d}t, \qquad \lambda_1 = \int_0^T \int_0^t (g_s \langle a_t, f_t\rangle + e_t \langle a_s, h_s\rangle)\mathrm{d}s\mathrm{d}t,$$
$$\lambda_2 = \int_0^T \int_0^t \langle a_t, f_t\rangle \langle a_s, h_s\rangle \mathrm{d}s\mathrm{d}t.$$

PROOF. The quantity on the left hand side is the summation of the four following quantities:

$$A_1 = \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t) \int_0^T e_t(\int_0^t g_s \mathrm{d}s)\mathrm{d}t], \quad A_2 = \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t) \int_0^T (\int_0^t g_s \mathrm{d}s) f_t \mathrm{d}W_t],$$
$$A_3 = \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t) \int_0^T e_t(\int_0^t h_s \mathrm{d}W_s)\mathrm{d}t], \quad A_4 = \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t) \int_0^T (\int_0^t h_s \mathrm{d}W_s) f_t \mathrm{d}W_t].$$

Since e and g are deterministic, the first term is equal to $A_1 = \lambda_0 \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t + \epsilon)]|_{\epsilon=0}$. Using Lemma A.1, one gets

$$A_2 = \Big(\int_0^T \langle a_t, f_t\rangle (\int_0^t g_s \mathrm{d}s)\mathrm{d}t\Big) \partial_\epsilon^1 \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t + \epsilon)]|_{\epsilon=0}.$$

First, write $A_3 = \int_0^T e_t \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t)(\int_0^t h_s \mathrm{d}W_s)]\mathrm{d}t$ and then, apply Lemma A.1 to obtain

$$A_3 = \Big(\int_0^T e_t(\int_0^t \langle a_s, h_s\rangle \mathrm{d}s)\mathrm{d}t\Big) \partial_\epsilon^1 \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t + \epsilon)]|_{\epsilon=0}.$$

Analogously, we show that $A_4 = \lambda_2 \partial_\epsilon^2 \mathbb{E}[l(\int_0^T a_t \mathrm{d}W_t + \epsilon)]|_{\epsilon=0}$. We are done. $\square$

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